Cross product and determinants (Sect. 12.4)

- Two definitions for the cross product.
- Geometric definition of cross product.
- Properties of the cross product.
- Cross product in vector components.
- Determinants to compute cross products.
- Triple product and volumes.

Two main ways to introduce the cross product

Geometrical definition	$\longrightarrow$	Properties	$\longrightarrow$	Expression in components.
Definition in components	$\longrightarrow$	Properties	$\longrightarrow$	Geometrical expression.

We choose the first way, like the textbook.

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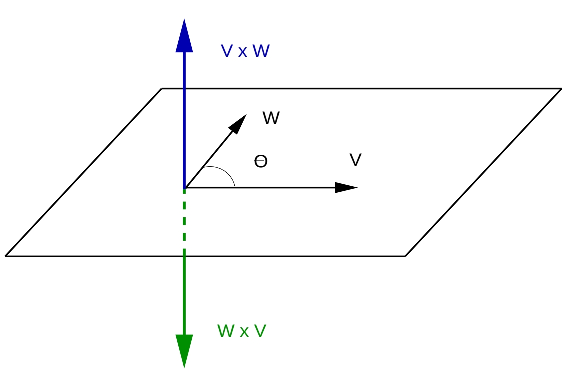
Geometric definition of cross product

Definition

The *cross product* of vectors $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^3$ having magnitudes $|\mathbf{v}|$, $|\mathbf{w}|$ and angle in between θ, where $0 \leq \theta \leq \pi$, is denoted by $\mathbf{v} \times \mathbf{w}$ and is the vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the right-hand rule, with norm

$$|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|\sin(\theta).$$

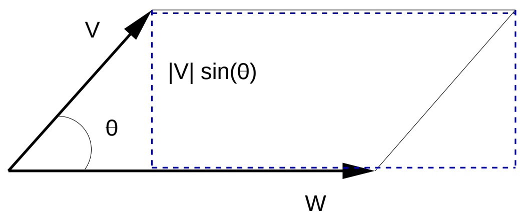


Remark: Cross product of two vectors is another vector; which is perpendicular to the original vectors.

Geometric definition of cross product

Theorem
$|\mathbf{v} \times \mathbf{w}|$ is the area of the parallelogram formed by vectors $\mathbf{v}$ and $\mathbf{w}$.

Proof.

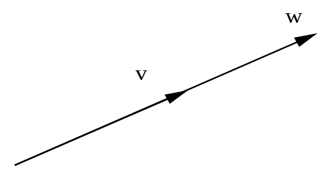


The area A of the parallelogram formed by $\mathbf{v}$ and $\mathbf{w}$ is

$$A = |\mathbf{w}|\big(|\mathbf{v}|\,\sin(\theta)\big) = |\mathbf{v} \times \mathbf{w}|.$$

□

Definition
Two vectors are *parallel* iff the angle in between them is $\theta = 0$.

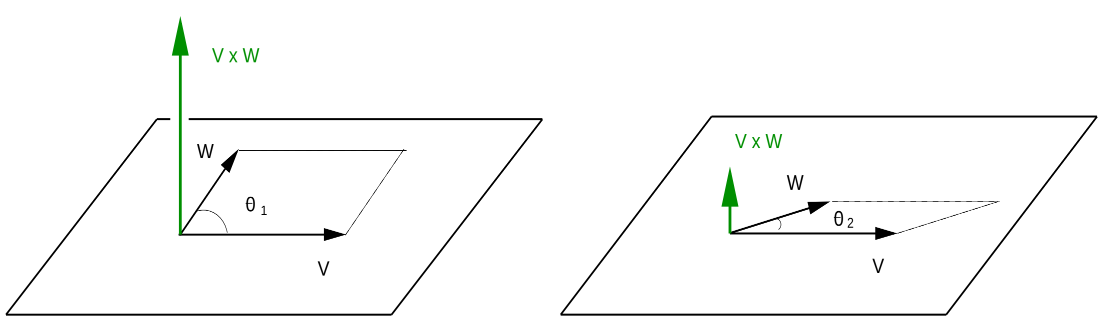


Theorem
The non-zero vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel iff $\mathbf{v} \times \mathbf{w} = \mathbf{0}$.

Geometric definition of cross product

Recall: $|\mathbf{v} \times \mathbf{w}|$ is the area of a parallelogram.

Example
The closer the vectors $\mathbf{v}$, $\mathbf{w}$ are to be parallel, the smaller is the area of the parallelogram they form, hence the shorter is their cross product vector $\mathbf{v} \times \mathbf{w}$.



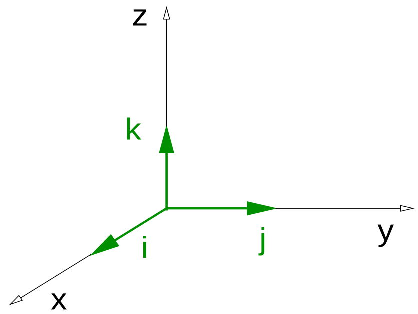
◁

Geometric definition of cross product

Example

Compute all cross products involving the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

Solution: Recall: $\mathbf{i} = \langle 1, 0, 0 \rangle$, $\mathbf{j} = \langle 0, 1, 0 \rangle$, $\mathbf{k} = \langle 0, 0, 1 \rangle$.



$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j},$$
$$\mathbf{i} \times \mathbf{i} = \mathbf{0}, \qquad \mathbf{j} \times \mathbf{j} = \mathbf{0}, \qquad \mathbf{k} \times \mathbf{k} = \mathbf{0},$$
$$\mathbf{i} \times \mathbf{k} = -\mathbf{j}, \qquad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}.$$

◁

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Properties of the cross product

Theorem

(a) $\mathbf{v} \times \mathbf{w} = -(\mathbf{w} \times \mathbf{v})$, *(skew-symmetric)*;

(b) $\mathbf{v} \times \mathbf{v} = \mathbf{0}$;

(c) $(a\mathbf{v}) \times \mathbf{w} = \mathbf{v} \times (a\mathbf{w}) = a(\mathbf{v} \times \mathbf{w})$, *(linear)*;

(d) $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$, *(linear)*;

(e) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$, *(not associative)*.

Proof.

Part (a) results from the right-hand rule and (b) from part (a). Parts (b) and (c) are proven in a similar ways as the linear property of the dot product. Part (d) is proven by giving an example. □

Properties of the cross product

Example

Show that the cross product is *not associative*, that is, $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$.

Solution: We prove this statement giving an example. We now show that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. Indeed,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = \mathbf{i} \times (-\mathbf{j}) = -(\mathbf{i} \times \mathbf{j}) = -\mathbf{k} \quad \Rightarrow \quad \mathbf{i} \times (\mathbf{i} \times \mathbf{k}) = -\mathbf{k},$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0} \times \mathbf{j} = \mathbf{0} \quad \Rightarrow \quad (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}.$$

We conclude that $\mathbf{i} \times (\mathbf{i} \times \mathbf{k}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{k} = \mathbf{0}$. ◁

Recall: The cross product of parallel vectors vanishes.

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Cross product in vector components

Theorem

The cross product of vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ *is given by*

$$\mathbf{v} \times \mathbf{w} = \langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \rangle.$$

Proof: Use the cross product properties and recall the non-zero cross products $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.
Express $\mathbf{v} = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$ and $\mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}$, then

$$\mathbf{v} \times \mathbf{w} = (v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}) \times (w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k}).$$

Use the linearity property. The only non-zero terms involve $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, and $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ and the symmetric analogues. The result is

$$\mathbf{v} \times \mathbf{w} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} + (v_3 w_1 - v_1 w_3)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

□

Cross product in vector components.

Example

Find $\mathbf{v} \times \mathbf{w}$ for $\mathbf{v} = \langle 1, 2, 0 \rangle$ and $\mathbf{w} = \langle 3, 2, 1 \rangle$,

Solution: We use the formula

$$\mathbf{v} \times \mathbf{w} = \big\langle (v_2 w_3 - v_3 w_2), (v_3 w_1 - v_1 w_3), (v_1 w_2 - v_2 w_1) \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \big\langle [(2)(1) - (0)(2)], [(0)(3) - (1)(1)], [(1)(2) - (2)(3)] \big\rangle$$

$$\mathbf{v} \times \mathbf{w} = \langle (2 - 0), (-1), (2 - 6) \rangle \quad \Rightarrow \quad \mathbf{v} \times \mathbf{w} = \langle 2, -1, -4 \rangle.$$

$\lhd$

Exercise: Find the angle between $\mathbf{v}$ and $\mathbf{w}$ above, using both the cross and the dot products. Verify that you get the same answer.

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Determinants to compute cross products.

Remark: Determinants help remember the $\mathbf{v} \times \mathbf{w}$ components.

Recall:

(a) The determinant of a 2×2 matrix is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

(b) The determinant of a 3×3 matrix is given by

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

2×2 determinants are used to find 3×3 determinants.

Determinants to compute cross products.

Theorem

The formula to compute determinants of 3×3 matrices can be used to find the the cross product $\mathbf{v} \times \mathbf{w}$, where $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, as follows

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Proof: Indeed, a straightforward computation shows that

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = (v_2 w_3 - v_3 w_2)\,\mathbf{i} - (v_1 w_3 - v_3 w_1)\,\mathbf{j} + (v_1 w_2 - v_2 w_1)\,\mathbf{k}.$$

□

Determinants to compute cross products.

Example

Given the vectors $\mathbf{v} = \langle 1, 2, 3 \rangle$ and $\mathbf{w} = \langle -2, 3, 1 \rangle$, compute both $\mathbf{w} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{w}$.

Solution: We need to compute the following determinant:

$$\mathbf{w} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ w_1 & w_2 & w_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

The result is

$$\mathbf{w} \times \mathbf{v} = (9-2)\,\mathbf{i} - (-6-1)\,\mathbf{j} + (-4-3)\,\mathbf{k} \quad \Rightarrow \quad \mathbf{w} \times \mathbf{v} = \langle 7, 7, -7 \rangle.$$

The properties of the determinant imply $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.
Hence, $\mathbf{v} \times \mathbf{w} = \langle -7, -7, 7 \rangle$. $\triangleleft$

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Triple product and volumes

Definition

The *triple product* of the vectors **u**, **v**, **w**, is the scalar $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.

Remarks:

(a) The triple product of three vectors is a scalar.

(b) The parentheses are important. First do the cross product, and only then dot the resulting vector with the first vector.

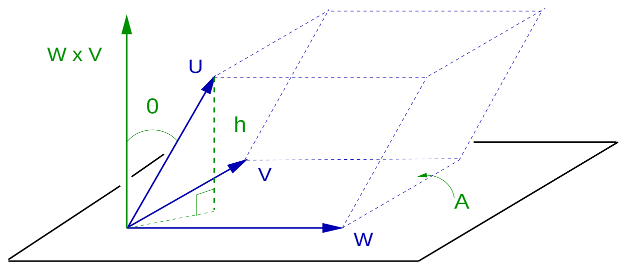
Theorem (Cyclic rotation formula for triple product)

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}).$$

Triple product and volumes

Theorem

The number $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$ *is the volume of the parallelepiped determined by the vectors* **u**, **v**, **w**.



Proof: Recall the dot product: $\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}|\,|\mathbf{y}|\cos(\theta)$. Then,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = |\mathbf{u}|\,|\mathbf{v} \times \mathbf{w}|\,|\cos(\theta)| = h\,|\mathbf{v} \times \mathbf{w}|.$$

$|\mathbf{v} \times \mathbf{w}|$ is the area A of the parallelogram formed by **v** and **w**. So,

$$|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| = h\,A,$$

which is the volume of the parallelepiped formed by **u**, **v**, **w**. $\square$

The triple product and volumes

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution: We use the formula $V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. We must compute the cross product first:

$$\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (2+2)\,\mathbf{i} - (3-1)\,\mathbf{j} + (-6-2)\,\mathbf{k},$$

that is, $\mathbf{v} \times \mathbf{w} = \langle 4, -2, -8 \rangle$. Now compute the dot product,

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \langle 1, 2, 3 \rangle \cdot \langle 4, -2, -8 \rangle = 4 - 4 - 24,$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = -24$. We conclude that $V = 24$. $\triangleleft$

The triple product and volumes

Remark: The triple product can be computed with a determinant.

Theorem

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, *and* $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$, *then*

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Example

Compute the volume of the parallelepiped formed by the vectors $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 3, 2, 1 \rangle$, $\mathbf{w} = \langle 1, -2, 1 \rangle$.

Solution:

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & -2 & 1 \end{vmatrix} = (1)(2+2) - (2)(3-1) + (3)(-6-2),$$

that is, $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = 4 - 4 - 24 = -24$. Hence $V = 24$. $\triangleleft$