Background knowledge

Contents:

This chapter contains material that is assumed knowledge for the course. It does not cover all assumed knowledge, as other necessary work is revised within the chapters.

A SURDS AND RADICALS

A **radical** is any number which is written with the **radical sign** $\sqrt{}$.

A **surd** is a real, irrational radical such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ or $\sqrt{6}$. Surds are present in solutions to some quadratic equations. $\sqrt{4}$ is a radical but is not a surd as it simplifies to 2.

$\sqrt{a}$ is the non-negative number such that $\boldsymbol{\sqrt{a} \times \sqrt{a} = a}$.

Properties:
- $\sqrt{a}$ is never negative, so $\boldsymbol{\sqrt{a} \geqslant 0}$.
- $\sqrt{a}$ is meaningful only for $a \geqslant 0$.
- $\boldsymbol{\sqrt{ab} = \sqrt{a} \times \sqrt{b}}$ for $a \geqslant 0$ and $b \geqslant 0$.
- $\boldsymbol{\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}}$ for $a \geqslant 0$ and $b > 0$.

Example 1 Self Tutor

Write as a single surd: **a** $\sqrt{2} \times \sqrt{3}$ **b** $\frac{\sqrt{18}}{\sqrt{6}}$

a $\sqrt{2} \times \sqrt{3}$
$= \sqrt{2 \times 3}$
$= \sqrt{6}$

b $\frac{\sqrt{18}}{\sqrt{6}}$
$= \sqrt{\frac{18}{6}}$
$= \sqrt{3}$

or $\frac{\sqrt{18}}{\sqrt{6}}$
$= \frac{\sqrt{6} \times \sqrt{3}}{\sqrt{6}}$
$= \sqrt{3}$

EXERCISE A

1 Write as a single surd or rational number:

a $\sqrt{3} \times \sqrt{5}$ **b** $(\sqrt{3})^2$ **c** $2\sqrt{2} \times \sqrt{2}$ **d** $3\sqrt{2} \times 2\sqrt{2}$

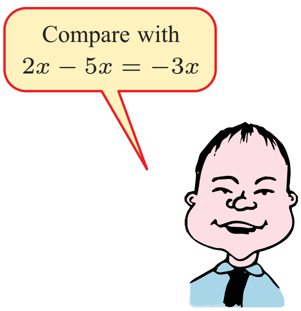
e $3\sqrt{7} \times 2\sqrt{7}$ **f** $\frac{\sqrt{12}}{\sqrt{2}}$ **g** $\frac{\sqrt{12}}{\sqrt{6}}$ **h** $\frac{\sqrt{18}}{\sqrt{3}}$

Example 2 Self Tutor

Simplify: **a** $3\sqrt{3} + 5\sqrt{3}$ **b** $2\sqrt{2} - 5\sqrt{2}$

a $3\sqrt{3} + 5\sqrt{3}$
$= (3 + 5)\sqrt{3}$
$= 8\sqrt{3}$

b $2\sqrt{2} - 5\sqrt{2}$
$= (2 - 5)\sqrt{2}$
$= -3\sqrt{2}$



2 Simplify the following mentally:

a $2\sqrt{2}+3\sqrt{2}$	**b** $2\sqrt{2}-3\sqrt{2}$	**c** $5\sqrt{5}-3\sqrt{5}$	**d** $5\sqrt{5}+3\sqrt{5}$
e $3\sqrt{5}-5\sqrt{5}$	**f** $7\sqrt{3}+2\sqrt{3}$	**g** $9\sqrt{6}-12\sqrt{6}$	**h** $\sqrt{2}+\sqrt{2}+\sqrt{2}$

Example 3 Self Tutor

Write $\sqrt{18}$ in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible.

$\sqrt{18}$
$=\sqrt{9\times 2}$ {9 is the largest perfect square factor of 18}
$=\sqrt{9}\times\sqrt{2}$
$=3\sqrt{2}$

3 Write the following in the form $a\sqrt{b}$ where a and b are integers and a is as large as possible:

a $\sqrt{8}$	**b** $\sqrt{12}$	**c** $\sqrt{20}$	**d** $\sqrt{32}$
e $\sqrt{27}$	**f** $\sqrt{45}$	**g** $\sqrt{48}$	**h** $\sqrt{54}$
i $\sqrt{50}$	**j** $\sqrt{80}$	**k** $\sqrt{96}$	**l** $\sqrt{108}$

Example 4 Self Tutor

Simplify: $2\sqrt{75}-5\sqrt{27}$

$$\begin{aligned}&2\sqrt{75}-5\sqrt{27}\\&=2\sqrt{25\times 3}-5\sqrt{9\times 3}\\&=2\times 5\times\sqrt{3}-5\times 3\times\sqrt{3}\\&=10\sqrt{3}-15\sqrt{3}\\&=-5\sqrt{3}\end{aligned}$$

4 Simplify:

a $4\sqrt{3}-\sqrt{12}$	**b** $3\sqrt{2}+\sqrt{50}$	**c** $3\sqrt{6}+\sqrt{24}$
d $2\sqrt{27}+2\sqrt{12}$	**e** $\sqrt{75}-\sqrt{12}$	**f** $\sqrt{2}+\sqrt{8}-\sqrt{32}$

Example 5 Self Tutor

Write $\frac{9}{\sqrt{3}}$ without a radical in the denominator.

$$\begin{aligned}&\frac{9}{\sqrt{3}}\\&=\frac{9}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\frac{9\sqrt{3}}{3}\\&=3\sqrt{3}\end{aligned}$$

5 Write without a radical in the denominator:

a $\frac{1}{\sqrt{2}}$ b $\frac{6}{\sqrt{3}}$ c $\frac{7}{\sqrt{2}}$ d $\frac{10}{\sqrt{5}}$

e $\frac{10}{\sqrt{2}}$ f $\frac{18}{\sqrt{6}}$ g $\frac{12}{\sqrt{3}}$ h $\frac{5}{\sqrt{7}}$

i $\frac{14}{\sqrt{7}}$ j $\frac{2\sqrt{3}}{\sqrt{2}}$

B SCIENTIFIC NOTATION (STANDARD FORM)

Scientific notation (or **standard form**) involves writing any given number as a number between 1 and 10, multiplied by a power of 10,

i.e., $a \times 10^k$ where $1 \leqslant a < 10$ and $k \in \mathbb{Z}$.

Example 6 Self Tutor

Write in standard form: a 37 600 b 0.000 86

a $37\,600 = 3.76 \times 10\,000$ {shift decimal point 4 places to the left and $\times$ 10 000}
$= 3.76 \times 10^4$

b $0.000\,86 = 8.6 \div 10^4$ {shift decimal point 4 places to the right and $\div$ 10 000}
$= 8.6 \times 10^{-4}$

EXERCISE B

1 Express the following in scientific notation:

a 259 b 259 000 c 2.59 d 0.259

e 0.000 259 f 40.7 g 4070 h 0.0407

i 407 000 j 407 000 000 k 0.000 040 7

2 Express the following in scientific notation:

a The distance from the Earth to the Sun is 149 500 000 000 m.

b Bacteria are single cell organisms, some of which have a diameter of 0.0003 mm.

c A speck of dust has width smaller than 0.001 mm.

d The core temperature of the Sun is 15 million degrees Celsius.

e A single red blood cell lives for about four months. During this time it will circulate around the body 300 000 times.

Example 7 **Self Tutor**

Write as an ordinary number:

a 3.2×10^2 **b** 5.76×10^{-5}

a 3.2×10^2
$= 3.20 \times 100$
$= 320$

b 5.76×10^{-5}
$= 000005.76 \div 10^5$
$= 0.000\,057\,6$

3 Write as an ordinary decimal number:

a 4×10^3 **b** 5×10^2 **c** 2.1×10^3 **d** 7.8×10^4
e 3.8×10^5 **f** 8.6×10^1 **g** 4.33×10^7 **h** 6×10^7

4 Write as an ordinary decimal number:

a 4×10^{-3} **b** 5×10^{-2} **c** 2.1×10^{-3} **d** 7.8×10^{-4}
e 3.8×10^{-5} **f** 8.6×10^{-1} **g** 4.33×10^{-7} **h** 6×10^{-7}

5 Write as an ordinary decimal number:

a The wavelength of light is 9×10^{-7} m.
b The estimated world population for the year 2000 was 6.130×10^9.
c The diameter of our galaxy, the Milky Way, is 1×10^5 light years.
d The smallest viruses are 1×10^{-5} mm in size.

6 Find, correct to 2 decimal places:

a $(3.42 \times 10^5) \times (4.8 \times 10^4)$ **b** $(6.42 \times 10^{-2})^2$ **c** $\dfrac{3.16 \times 10^{-10}}{6 \times 10^7}$

d $(9.8 \times 10^{-4}) \div (7.2 \times 10^{-6})$ **e** $\dfrac{1}{3.8 \times 10^5}$ **f** $(1.2 \times 10^3)^3$

7 If a missile travels at 5400 km h^{-1}, how far will it travel in:

a 1 day **b** 1 week **c** 2 years?

Give your answers in scientific notation correct to 2 decimal places, and assume that 1 year $\approx$ 365.25 days.

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8 Light travels at a speed of 3×10^8 metres per second. How far will light travel in:

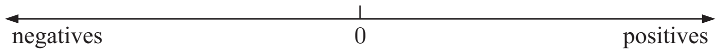
a 1 minute **b** 1 day **c** 1 year?

Give your answers with decimal part correct to 2 decimal places, and assume that 1 year $\approx$ 365.25 days.

C NUMBER SYSTEMS AND SET NOTATION

NUMBER SYSTEMS

We use:
- $\mathbb{R}$ to represent the set of all **real numbers**. These include all of the numbers on the number line.



- $\mathbb{N}$ to represent the set of all **natural numbers**. $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$
- $\mathbb{Z}$ to represent the set of all **integers**. $\mathbb{Z} = \{0, \pm1, \pm2, \pm3, \pm4, ...\}$
- $\mathbb{Z}^+$ is the set of all positive integers. $\mathbb{Z}^+ = \{1, 2, 3, 4, ...\}$
- $\mathbb{Q}$ to represent the set of all **rational numbers** which are any numbers of the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$.

SET NOTATION

$\{x \,|\, -3 < x < 2\}$ reads "the set of all values that x can be such that x lies between -3 and 2".

the set of all *such that*

Unless stated otherwise, we assume that x is real.

EXERCISE C

1 Write verbal statements for the meaning of:

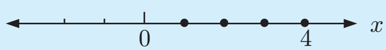
a $\{x \,|\, x > 5,\ x \in \mathbb{R}\}$ **b** $\{x \,|\, x \leqslant 3,\ x \in \mathbb{Z}\}$ **c** $\{y \,|\, 0 < y < 6\}$

d $\{x \,|\, 2 \leqslant x \leqslant 4,\ x \in \mathbb{Z}\}$ **e** $\{t \,|\, 1 < t < 5\}$ **f** $\{n \,|\, n < 2 \text{ or } n \geqslant 6\}$

Example 8 **Self Tutor**

Write in set notation:

a



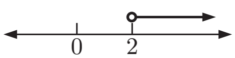
b

a $\{x \,|\, 1 \leqslant x \leqslant 4,\quad x \in \mathbb{N}\}$
or $\{x \,|\, 1 \leqslant x \leqslant 4,\quad x \in \mathbb{Z}\}$

b $\{x \,|\, -3 \leqslant x < 4,\ x \in \mathbb{R}\}$

2 Write in set notation:

a



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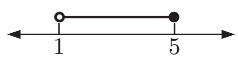

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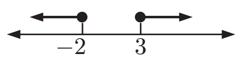
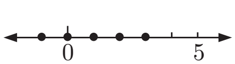
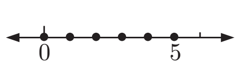


d



e



f

3 Sketch the following number sets:

a $\{x \mid 4 \leqslant x < 10,\ x \in \mathbb{N}\}$ **b** $\{x \mid -4 < x \leqslant 5,\ x \in \mathbb{Z}\}$

c $\{x \mid -5 < x \leqslant 4,\ x \in \mathbb{R}\}$ **d** $\{x \mid x > -4,\ x \in \mathbb{Z}\}$

e $\{x \mid x \leqslant 8,\ x \in \mathbb{R}\}$

D ALGEBRAIC SIMPLIFICATION

To answer the following questions, you will need to remember:

- the distributive law $\boldsymbol{a(b+c) = ab + ac}$
- power notation $a^2 = a \times a,\ a^3 = a \times a \times a,\ a^4 = a \times a \times a \times a,$ and so on.

EXERCISE D

1 Simplify if possible:

a $3x + 7x - 10$ **b** $3x + 7x - x$ **c** $2x + 3x + 5y$

d $8 - 6x - 2x$ **e** $7ab + 5ba$ **f** $3x^2 + 7x^3$

2 Remove the brackets and then simplify:

a $3(2x+5) + 4(5+4x)$ **b** $6 - 2(3x-5)$

c $5(2a-3b) - 6(a-2b)$ **d** $3x(x^2 - 7x + 3) - (1 - 2x - 5x^2)$

3 Simplify:

a $2x(3x)^2$ **b** $\dfrac{3a^2b^3}{9ab^4}$ **c** $\sqrt{16x^4}$ **d** $(2a^2)^3 \times 3a^4$

E LINEAR EQUATIONS AND INEQUALITIES

When dealing with inequalities:

- multiplying or dividing both sides by a negative reverses the inequality sign.
- do not multiply or divide both sides by the unknown or a term involving the unknown.

EXERCISE E

1 Solve for x:

a $2x + 5 = 25$ **b** $3x - 7 > 11$ **c** $5x + 16 = 20$

d $\dfrac{x}{3} - 7 = 10$ **e** $6x + 11 < 4x - 9$ **f** $\dfrac{3x-2}{5} = 8$

g $1 - 2x \geqslant 19$ **h** $\frac{1}{2}x + 1 = \frac{2}{3}x - 2$ **i** $\frac{2}{3} - \dfrac{3x}{4} = \frac{1}{2}(2x-1)$

2 Solve simultaneously for x and y:

a $\begin{cases} x + 2y = 9 \\ x - y = 3 \end{cases}$ **b** $\begin{cases} 2x + 5y = 28 \\ x - 2y = 2 \end{cases}$ **c** $\begin{cases} 7x + 2y = -4 \\ 3x + 4y = 14 \end{cases}$

d $\begin{cases} 5x - 4y = 27 \\ 3x + 2y = 9 \end{cases}$ **e** $\begin{cases} x + 2y = 5 \\ 2x + 4y = 1 \end{cases}$ **f** $\begin{cases} \dfrac{x}{2} + \dfrac{y}{3} = 5 \\ \dfrac{x}{3} + \dfrac{y}{4} = 1 \end{cases}$

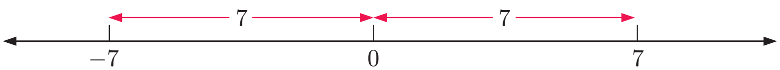
F MODULUS OR ABSOLUTE VALUE

The **modulus** or **absolute value** of a real number is its size, ignoring its sign.

For example: the modulus of 7 is 7, and
the modulus of -7 is also 7.

GEOMETRIC DEFINITION

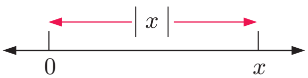
The modulus of a real number is its *distance* from zero on the number line. Because the modulus is a distance, it cannot be negative.



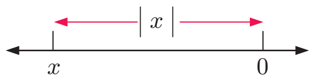
We denote the modulus of x as $|x|$.

$|x|$ is the distance of x from 0 on the real number line.

If $x > 0$



If $x < 0$



$|x - a|$ can be considered as 'the distance of x from a'.

ALGEBRAIC DEFINITION

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{or} \quad |x| = \sqrt{x^2}$$

MODULUS EQUATIONS

It is clear that $|x| = 2$ has two solutions, $x = 2$ and $x = -2$, since $|2| = 2$ and $|-2| = 2$.

If $|x| = a$ where $a > 0$, then $x = \pm a$.

EXERCISE F

1 Find the value of:

a $5-(-11)$ **b** $|5|-|-11|$ **c** $|5-(-11)|$

d $\left|(-2)^2+11(-2)\right|$ **e** $|-6|-|-8|$ **f** $|-6-(-8)|$

2 If $a=-2$, $b=3$, and $c=-4$ find the value of:

a $|a|$ **b** $|b|$ **c** $|a|\,|b|$ **d** $|ab|$

e $|a-b|$ **f** $|a|-|b|$ **g** $|a+b|$ **h** $|a|+|b|$

i $|a|^2$ **j** a^2 **k** $\left|\dfrac{c}{a}\right|$ **l** $\dfrac{|c|}{|a|}$

3 Solve for x:

a $|x|=3$ **b** $|x|=-5$ **c** $|x|=0$

d $|x-1|=3$ **e** $|3-x|=4$ **f** $|x+5|=-1$

g $|3x-2|=1$ **h** $|3-2x|=3$ **i** $|2-5x|=12$

G PRODUCT EXPANSION

$y=2(x-1)(x+3)$ can be expanded into the general form $y=ax^2+bx+c$.

Likewise, $y=2(x-3)^2+7$ can also be expanded into this form.

Following is a **list of expansion rules** you can use:

- $\boldsymbol{(a+b)(c+d)=ac+ad+bc+bd}$
- $\boldsymbol{(a+b)(a-b)=a^2-b^2}$ {**difference of two squares**}
- $\boldsymbol{(a+b)^2=a^2+2ab+b^2}$ {**perfect squares**}

Example 9 **Self Tutor**

Expand and simplify:

a $(2x+1)(x+3)$ **b** $(3x-2)(x+3)$

a
$$\begin{aligned}&(2x+1)(x+3)\\&=2x^2+6x+x+3\\&=2x^2+7x+3\end{aligned}$$

b
$$\begin{aligned}&(3x-2)(x+3)\\&=3x^2+9x-2x-6\\&=3x^2+7x-6\end{aligned}$$

EXERCISE G

1 Expand and simplify using $(a+b)(c+d) = ac + ad + bc + bd$:

a $(2x+3)(x+1)$ **b** $(3x+4)(x+2)$ **c** $(5x-2)(2x+1)$

d $(x+2)(3x-5)$ **e** $(7-2x)(2+3x)$ **f** $(1-3x)(5+2x)$

g $(3x+4)(5x-3)$ **h** $(1-3x)(2-5x)$ **i** $(7-x)(3-2x)$

j $(5-2x)(3-2x)$ **k** $-(x+1)(x+2)$ **l** $-2(x-1)(2x+3)$

Example 10 Self Tutor

Expand using the rule $(a+b)(a-b) = a^2 - b^2$:

a $(5x-2)(5x+2)$ **b** $(7+2x)(7-2x)$

a $(5x-2)(5x+2)$
$= (5x)^2 - 2^2$
$= 25x^2 - 4$

b $(7+2x)(7-2x)$
$= 7^2 - (2x)^2$
$= 49 - 4x^2$

2 Expand using the rule $(a+b)(a-b) = a^2 - b^2$:

a $(x+6)(x-6)$ **b** $(x+8)(x-8)$ **c** $(2x-1)(2x+1)$

d $(3x-2)(3x+2)$ **e** $(4x+5)(4x-5)$ **f** $(5x-3)(5x+3)$

g $(3-x)(3+x)$ **h** $(7-x)(7+x)$ **i** $(7+2x)(7-2x)$

j $(x+\sqrt{2})(x-\sqrt{2})$ **k** $(x+\sqrt{5})(x-\sqrt{5})$ **l** $(2x-\sqrt{3})(2x+\sqrt{3})$

Example 11 Self Tutor

Expand using the perfect square expansion rule:

a $(x+2)^2$ **b** $(3x-1)^2$

a $(x+2)^2$
$= x^2 + 2(x)(2) + 2^2$
$= x^2 + 4x + 4$

b $(3x-1)^2$
$= (3x)^2 + 2(3x)(-1) + (-1)^2$
$= 9x^2 - 6x + 1$

3 Expand and simplify using the perfect square expansion rule:

a $(x+5)^2$ **b** $(x+7)^2$ **c** $(x-2)^2$ **d** $(x-6)^2$

e $(3+x)^2$ **f** $(5+x)^2$ **g** $(11-x)^2$ **h** $(10-x)^2$

i $(2x+7)^2$ **j** $(3x+2)^2$ **k** $(5-2x)^2$ **l** $(7-3x)^2$

4 Expand the following into the general form $y = ax^2 + bx + c$:

a $y = 2(x+2)(x+3)$ **b** $y = 3(x-1)^2 + 4$ **c** $y = -(x+1)(x-7)$

d $y = -(x+2)^2 - 11$ **e** $y = 4(x-1)(x-5)$ **f** $y = -\frac{1}{2}(x+4)^2 - 6$

g $y = -5(x-1)(x-6)$ **h** $y = \frac{1}{2}(x+2)^2 - 6$ **i** $y = -\frac{5}{2}(x-4)^2$

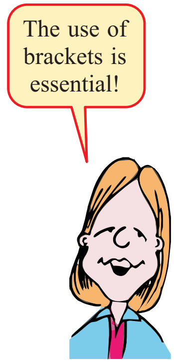
Example 12 — Self Tutor

Expand and simplify:

a $1-2(x+3)^2$ **b** $2(3+x)-(2+x)(3-x)$

a
$$\begin{aligned} &1-2(x+3)^2 \\ &= 1-2[x^2+6x+9] \\ &= 1-2x^2-12x-18 \\ &= -2x^2-12x-17 \end{aligned}$$

b
$$\begin{aligned} &2(3+x)-(2+x)(3-x) \\ &= 6+2x-[6-2x+3x-x^2] \\ &= 6+2x-6+2x-3x+x^2 \\ &= x^2+x \end{aligned}$$



5 Expand and simplify:

a $1+2(x+3)^2$
b $2+3(x-2)(x+3)$
c $3-(3-x)^2$
d $5-(x+5)(x-4)$
e $1+2(4-x)^2$
f $x^2-3x-(x+2)(x-2)$
g $(x+2)^2-(x+1)(x-4)$
h $(2x+3)^2+3(x+1)^2$
i $x^2+3x-2(x-4)^2$
j $(3x-2)^2-2(x+1)^2$

H FACTORISATION

Algebraic **factorisation** is the reverse process of expansion.

For example, $(2x+1)(x-3)$ is **expanded** to $2x^2-5x-3$, whereas
$2x^2-5x-3$ is **factorised** to $(2x+1)(x-3)$.

Notice that $2x^2-5x-3=(2x+1)(x-3)$ has been factorised into two **linear factors**.

Flow chart for factorising:

Expression

↓

Take out any common factors

↓

Recognise type

Difference of two squares
$a^2-b^2=(a+b)(a-b)$

Perfect square
$a^2+2ab+b^2=(a+b)^2$

Sum and Product type
$\boldsymbol{ax^2+bx+c, \quad a \neq 0}$

- find ac
- find the factors of ac which add to b
- if these factors are p and q, replace bx by $px+qx$
- complete the factorisation

Sum and Product type
$\boldsymbol{x^2+bx+c}$

$x^2+bx+c=(x+p)(x+q)$
where $p+q=b$ and $pq=c$

Example 13 Self Tutor

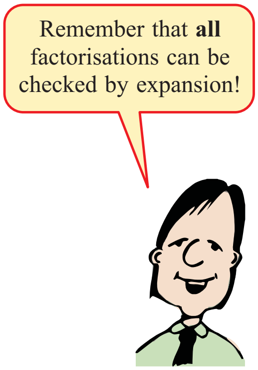
Fully factorise:

a $3x^2 - 12x$ **b** $4x^2 - 1$ **c** $x^2 - 12x + 36$

a $3x^2 - 12x$ {$3x$ is a common factor}
$= 3x(x - 4)$

b $4x^2 - 1$ {difference of two squares}
$= (2x)^2 - 1^2$
$= (2x + 1)(2x - 1)$

c $x^2 - 12x + 36$ {perfect square form}
$= x^2 + 2(x)(-6) + (-6)^2$
$= (x - 6)^2$



EXERCISE H

1 Fully factorise:

a $3x^2 + 9x$ **b** $2x^2 + 7x$ **c** $4x^2 - 10x$
d $6x^2 - 15x$ **e** $9x^2 - 25$ **f** $16x^2 - 1$
g $2x^2 - 8$ **h** $3x^2 - 9$ **i** $4x^2 - 20$
j $x^2 - 8x + 16$ **k** $x^2 - 10x + 25$ **l** $2x^2 - 8x + 8$
m $16x^2 + 40x + 25$ **n** $9x^2 + 12x + 4$ **o** $x^2 - 22x + 121$

Example 14 Self Tutor

Fully factorise:

a $3x^2 + 12x + 9$ **b** $-x^2 + 3x + 10$

a $3x^2 + 12x + 9$ {3 is a common factor}
$= 3(x^2 + 4x + 3)$ {sum $= 4$, product $= 3$}
$= 3(x + 1)(x + 3)$

b $-x^2 + 3x + 10$
$= -[x^2 - 3x - 10]$ {removing -1 as a common factor to make the coefficient of x^2 be 1}
$= -(x - 5)(x + 2)$ {sum $= -3$, product $= -10$}

2 Fully factorise:

a $x^2 + 9x + 8$ **b** $x^2 + 7x + 12$ **c** $x^2 - 7x - 18$
d $x^2 + 4x - 21$ **e** $x^2 - 9x + 18$ **f** $x^2 + x - 6$
g $-x^2 + x + 2$ **h** $3x^2 - 42x + 99$ **i** $-2x^2 - 4x - 2$

j $2x^2 + 6x - 20$ **k** $2x^2 - 10x - 48$ **l** $-2x^2 + 14x - 12$

m $-3x^2 + 6x - 3$ **n** $-x^2 - 2x - 1$ **o** $-5x^2 + 10x + 40$

FACTORISATION BY 'SPLITTING' THE x-TERM

Using the distributive law to expand we see that:
$$\begin{aligned} &(2x+3)(4x+5) \\ &= 8x^2 + 10x + 12x + 15 \\ &= 8x^2 + 22x + 15 \end{aligned}$$

We will now **reverse** the process to **factorise** the quadratic expression $8x^2 + 22x + 15$.

		$8x^2 + 22x + 15$
Step 1:	'Split' the middle term	$= 8x^2 + 10x + 12x + 15$
Step 2:	Group in pairs	$= (8x^2 + 10x) + (12x + 15)$
Step 3:	Factorise each pair separately	$= 2x(4x+5) + 3(4x+5)$
Step 4:	Factorise fully	$= (4x+5)(2x+3)$

The 'trick' in factorising these types of quadratic expressions is in *Step 1*. The middle term is 'split' into two so the rest of the factorisation can proceed smoothly.

Rules for splitting the x-term:

The following procedure is recommended for factorising $ax^2 + bx + c$:

- Find ac.
- Find the factors of ac which add to b.
- If these factors are p and q, replace bx by $px + qx$.
- Complete the factorisation.

Example 15 **Self Tutor**

Fully factorise:

a $2x^2 - x - 10$ **b** $6x^2 - 25x + 14$

a $2x^2 - x - 10$ has $ac = 2 \times -10 = -20$.
The factors of -20 which add to -1 are -5 and $+4$.
$$\begin{aligned} \therefore \quad 2x^2 - x - 10 &= 2x^2 - 5x + 4x - 10 \\ &= x(2x-5) + 2(2x-5) \\ &= (2x-5)(x+2) \end{aligned}$$

b $6x^2 - 25x + 14$ has $ac = 6 \times 14 = 84$.
The factors of 84 which add to -25 are -21 and -4.
$$\begin{aligned} \therefore \quad 6x^2 - 25x + 14 &= 6x^2 - 21x - 4x + 14 \\ &= 3x(2x-7) - 2(2x-7) \\ &= (2x-7)(3x-2) \end{aligned}$$

3 Fully factorise:

a $2x^2 + 5x - 12$ **b** $3x^2 - 5x - 2$ **c** $7x^2 - 9x + 2$
d $6x^2 - x - 2$ **e** $4x^2 - 4x - 3$ **f** $10x^2 - x - 3$
g $2x^2 - 11x - 6$ **h** $3x^2 - 5x - 28$ **i** $8x^2 + 2x - 3$
j $10x^2 - 9x - 9$ **k** $3x^2 + 23x - 8$ **l** $6x^2 + 7x + 2$
m $-4x^2 - 2x + 6$ **n** $12x^2 - 16x - 3$ **o** $-6x^2 - 9x + 42$
p $21x - 10 - 9x^2$ **q** $8x^2 - 6x - 27$ **r** $12x^2 + 13x + 3$
s $12x^2 + 20x + 3$ **t** $15x^2 - 22x + 8$ **u** $14x^2 - 11x - 15$

Example 16 Self Tutor

Fully factorise: $3(x+2) + 2(x-1)(x+2) - (x+2)^2$

$$\begin{aligned}
&3(x+2) + 2(x-1)(x+2) - (x+2)^2 \\
&= (x+2)[3 + 2(x-1) - (x+2)] \quad \{\text{as } (x+2) \text{ is a common factor}\} \\
&= (x+2)[3 + 2x - 2 - x - 2] \\
&= (x+2)(x-1)
\end{aligned}$$

4 Fully factorise:

a $3(x+4) + 2(x+4)(x-1)$ **b** $8(2-x) - 3(x+1)(2-x)$
c $6(x+2)^2 + 9(x+2)$ **d** $4(x+5) + 8(x+5)^2$
e $(x+2)(x+3) - (x+3)(2-x)$ **f** $(x+3)^2 + 2(x+3) - x(x+3)$
g $5(x-2) - 3(2-x)(x+7)$ **h** $3(1-x) + 2(x+1)(x-1)$

Example 17 Self Tutor

Fully factorise using the 'difference of two squares':

a $(x+2)^2 - 9$ **b** $(1-x)^2 - (2x+1)^2$

a
$$\begin{aligned}
&(x+2)^2 - 9 \\
&= (x+2)^2 - 3^2 \\
&= [(x+2)+3][(x+2)-3] \\
&= (x+5)(x-1)
\end{aligned}$$

b
$$\begin{aligned}
&(1-x)^2 - (2x+1)^2 \\
&= [(1-x)+(2x+1)][(1-x)-(2x+1)] \\
&= [1-x+2x+1][1-x-2x-1] \\
&= -3x(x+2)
\end{aligned}$$

5 Fully factorise:

a $(x+3)^2 - 16$ **b** $4 - (1-x)^2$ **c** $(x+4)^2 - (x-2)^2$
d $16 - 4(x+2)^2$ **e** $(2x+3)^2 - (x-1)^2$ **f** $(x+h)^2 - x^2$
g $3x^2 - 3(x+2)^2$ **h** $5x^2 - 20(2-x)^2$ **i** $12x^2 - 27(3+x)^2$

INVESTIGATION ANOTHER FACTORISATION TECHNIQUE

What to do:

1 By expanding, show that $\dfrac{(ax+p)(ax+q)}{a} = ax^2+[p+q]x+\left[\dfrac{pq}{a}\right]$.

2 If $ax^2+bx+c = \dfrac{(ax+p)(ax+q)}{a}$, show that $p+q=b$ and $pq=ac$.

3 Using **2** on $8x^2+22x+15$, we have

$$8x^2+22x+15 = \frac{(8x+p)(8x+q)}{8} \quad \text{where} \quad \begin{cases} p+q=22 \\ pq = 8\times 15 = 120 \end{cases}$$

$\therefore \quad p=12,\ q=10,$ or vice versa

$$\begin{aligned} \therefore \quad 8x^2+22x+15 &= \frac{(8x+12)(8x+10)}{8} \\ &= \frac{\cancel{4}(2x+3)\cancel{2}(4x+5)}{\cancel{8}_1} \\ &= (2x+3)(4x+5) \end{aligned}$$

a Use the method shown to factorise:

i $3x^2+14x+8$ **ii** $12x^2+17x+6$ **iii** $15x^2+14x-8$

b Check your answers to **a** by expansion.

I FORMULA REARRANGEMENT

In the formula $D = xt+p$ we say that D is the **subject**. This is because D is expressed in terms of the other variables x, t and p.

We can rearrange the formula to make one of the other variables the subject. We do this using the usual rules for solving equations. Whatever we do to one side of the equation we must also do to the other side.

Example 18 **Self Tutor**

Make x the subject of $D = xt+p$.

$$\begin{aligned} \text{If} \quad D &= xt+p \\ \therefore \quad xt+p &= D \\ \therefore \quad xt+p-p &= D-p \quad \{\text{subtracting } p \text{ from both sides}\} \\ \therefore \quad xt &= D-p \\ \therefore \quad \frac{xt}{t} &= \frac{D-p}{t} \\ \therefore \quad x &= \frac{D-p}{t} \quad \{\text{dividing both sides by } t\} \end{aligned}$$

EXERCISE I

1 Make x the subject of:

a $a+x=b$ b $ax=b$ c $2x+a=d$
d $c+x=t$ e $5x+2y=20$ f $2x+3y=12$
g $7x+3y=d$ h $ax+by=c$ i $y=mx+c$

Example 19 **Self Tutor**

Make z the subject of $c=\dfrac{m}{z}$.

$$c=\frac{m}{z}$$

$$c\times z=\frac{m}{z}\times z \qquad \{\text{multiplying both sides by } z\}$$

$$\therefore \quad cz=m$$

$$\therefore \quad \frac{cz}{c}=\frac{m}{c} \qquad \{\text{dividing both sides by } c\}$$

$$\therefore \quad z=\frac{m}{c}$$

2 Make z the subject of:

a $az=\dfrac{b}{c}$ b $\dfrac{a}{z}=d$ c $\dfrac{3}{d}=\dfrac{2}{z}$ d $\dfrac{z}{2}=\dfrac{a}{z}$

3 Make:

a a the subject of $F=ma$
b r the subject of $C=2\pi r$
c d the subject of $V=ldh$
d K the subject of $A=\dfrac{b}{K}$.

Example 20 **Self Tutor**

Make t the subject of $s=\frac{1}{2}gt^2$ where $t>0$.

$$\tfrac{1}{2}gt^2=s \qquad \{\text{writing with } t^2 \text{ on LHS}\}$$

$$\therefore \quad 2\times\tfrac{1}{2}gt^2=2\times s \qquad \{\text{multiplying both sides by } 2\}$$

$$\therefore \quad gt^2=2s$$

$$\therefore \quad \frac{gt^2}{g}=\frac{2s}{g} \qquad \{\text{dividing both sides by } g\}$$

$$\therefore \quad t^2=\frac{2s}{g}$$

$$\therefore \quad t=\sqrt{\frac{2s}{g}} \qquad \{\text{as } t>0\}$$

4 Make:

a r the subject of $A = \pi r^2$ if $r > 0$

b x the subject of $N = \dfrac{x^5}{a}$

c r the subject of $V = \frac{4}{3}\pi r^3$

d x the subject of $D = \dfrac{n}{x^3}$.

5 Make:

a a the subject of $d = \dfrac{\sqrt{a}}{n}$

b l the subject of $T = \frac{1}{5}\sqrt{l}$

c a the subject of $c = \sqrt{a^2 - b^2}$

d l the subject of $T = 2\pi\sqrt{\dfrac{l}{g}}$

e a the subject of $P = 2(a + b)$

f h the subject of $A = \pi r^2 + 2\pi rh$

g r the subject of $I = \dfrac{E}{R + r}$

h q the subject of $A = \dfrac{B}{p - q}$.

6 **a** Make a the subject of the formula $k = \dfrac{d^2}{2ab}$.

b Find the value for a when $k = 112$, $d = 24$, $b = 2$.

7 The formula for determining the volume of a sphere of radius r is $V = \frac{4}{3}\pi r^3$.

a Make r the subject of the formula.

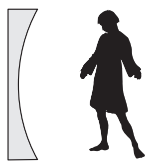
b Find the radius of a sphere which has a volume of 40 cm^3.

8 The distance travelled by an object accelerating from a stationary position is given by the formula $S = \frac{1}{2}at^2$ cm where a is the acceleration in cm s^{-2} and t is the time in seconds.

a Make t the subject of the formula. Consider $t > 0$ only.

b Find the time taken for an object accelerating at 8 cm s^{-2} to travel 10 m.

9 The relationship between the object and image distances (in cm) for a concave mirror can be written as $\dfrac{1}{f} = \dfrac{1}{u} + \dfrac{1}{v}$ where f is the focal length, u is the object distance and v is the image distance.



a Make v the subject of the formula.

b Given a focal length of 8 cm, find the image distance for the following object distances: **i** 50 cm **ii** 30 cm.

10 According to Einstein's theory of relativity, the mass of a particle is given by the formula $m = \dfrac{m_0}{\sqrt{1 - \left(\dfrac{v}{c}\right)^2}}$, where m_0 is the mass of the particle at rest, v is the speed of the particle, and c is the speed of light in a vacuum.

a Make v the subject of the formula given $v > 0$.

b Find the speed necessary to increase the mass of a particle to three times its rest mass, i.e., $m = 3m_0$. Give the value for v as a fraction of c.

c A cyclotron increased the mass of an electron to $30m_0$. With what velocity must the electron have been travelling, given $c = 3 \times 10^8$ m s^{-1}?

J ADDING AND SUBTRACTING ALGEBRAIC FRACTIONS

To add or subtract algebraic fractions, we combine them into a single fraction with the **least common denominator** (LCD).

For example, $\dfrac{x-1}{3} - \dfrac{x+3}{2}$ has LCD of 6, so we write each fraction with denominator 6.

Example 21 Self Tutor

Write as a single fraction: **a** $2 + \dfrac{3}{x}$ **b** $\dfrac{x-1}{3} - \dfrac{x+3}{2}$

a
$$\begin{aligned} &2 + \frac{3}{x} \\ &= 2\left(\frac{x}{x}\right) + \frac{3}{x} \\ &= \frac{2x+3}{x} \end{aligned}$$

b
$$\begin{aligned} &\frac{x-1}{3} - \frac{x+3}{2} \\ &= \frac{2}{2}\left(\frac{x-1}{3}\right) - \frac{3}{3}\left(\frac{x+3}{2}\right) \\ &= \frac{2(x-1) - 3(x+3)}{6} \\ &= \frac{2x-2-3x-9}{6} \\ &= \frac{-x-11}{6} \end{aligned}$$

EXERCISE J

1 Write as a single fraction:

a $3 + \dfrac{x}{5}$ **b** $1 + \dfrac{3}{x}$ **c** $3 + \dfrac{x-2}{2}$

d $3 - \dfrac{x-2}{4}$ **e** $\dfrac{2+x}{3} + \dfrac{x-4}{5}$ **f** $\dfrac{2x+5}{4} - \dfrac{x-1}{6}$

Example 22 Self Tutor

Write $\dfrac{3x+1}{x-2} - 2$ as a single fraction.

$$\begin{aligned} &\frac{3x+1}{x-2} - 2 \\ &= \left(\frac{3x+1}{x-2}\right) - 2\left(\frac{x-2}{x-2}\right) \quad \{\text{LCD} = (x-2)\} \\ &= \frac{(3x+1) - 2(x-2)}{x-2} \\ &= \frac{3x+1-2x+4}{x-2} \\ &= \frac{x+5}{x-2} \end{aligned}$$

2 Write as a single fraction:

a $1+\dfrac{3}{x+2}$ **b** $-2+\dfrac{3}{x-4}$ **c** $-3-\dfrac{2}{x-1}$

d $\dfrac{2x-1}{x+1}+3$ **e** $3-\dfrac{x}{x+1}$ **f** $-1+\dfrac{4}{1-x}$

3 Write as a single fraction:

a $\dfrac{3x}{2x-5}+\dfrac{2x+5}{x-2}$ **b** $\dfrac{1}{x-2}-\dfrac{1}{x-3}$

c $\dfrac{5x}{x-4}+\dfrac{3x-2}{x+4}$ **d** $\dfrac{2x+1}{x-3}-\dfrac{x+4}{2x+1}$

K CONGRUENCE AND SIMILARITY

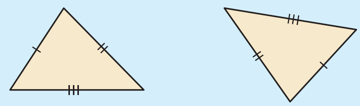
CONGRUENCE

Two triangles are **congruent** if they are identical in every respect apart from position. The triangles have the same shape and size.

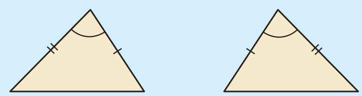
There are four acceptable tests for the **congruence of two triangles**.

Two triangles are congruent if one of the following is true:

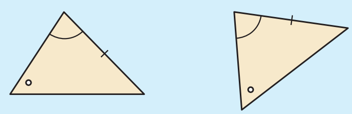
- corresponding sides are equal in length (**SSS**)



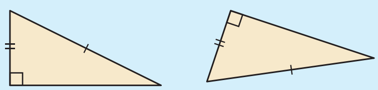
- two sides and the included angle are equal (**SAS**)



- two angles and a pair of corresponding sides are equal (**AAcorS**)



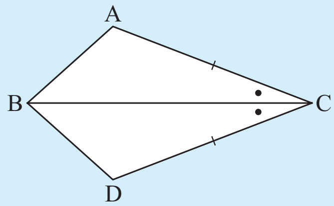
- for right angled triangles, the hypotenuse and one other pair of sides are equal (**RHS**).

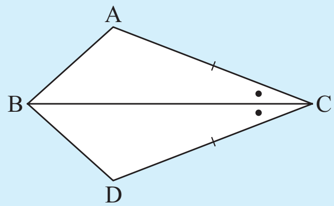


If congruence can be proven then all corresponding lengths, angles and areas must be equal.

Example 23 — Self Tutor

Explain why ΔABC and ΔDBC are congruent:



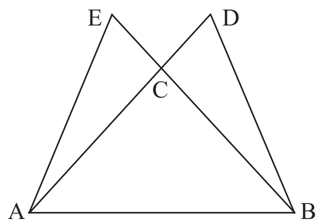


Δs ABC and DBC are congruent (SAS) as:

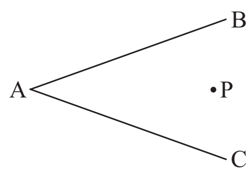
- AC = DC
- $A\widehat{C}B = D\widehat{C}B$, and
- [BC] is common to both.

EXERCISE K.1

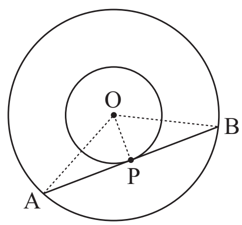
1 Triangle ABC is isosceles with AC = BC. [BC] and [AC] are produced to E and D respectively so that CE = CD. Prove that AE = BD.



2 Point P is equidistant from both [AB] and [AC]. Use congruence to show that P lies on the bisector of $B\widehat{A}C$.



3 Two concentric circles are drawn. At P on the inner circle, a tangent is drawn which meets the other circle at A and B. Use triangle congruence to prove that P is the midpoint of [AB].



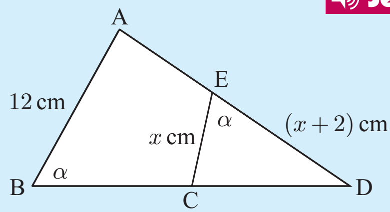
SIMILARITY

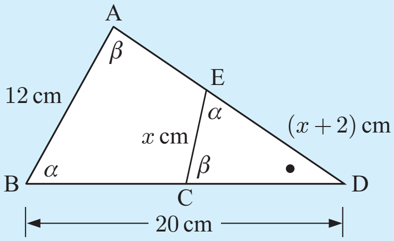
Two triangles are **similar** if one is an enlargement of the other.

Similar triangles are **equiangular**, and have corresponding sides in the same **ratio**.

Example 24 **Self Tutor**

Establish that a pair of triangles is similar, then find x given BD = 20 cm.





α	β	$\bullet$	
-	$x+2$	x	small Δ
-	20	12	large Δ

The triangles are equiangular and hence similar.

$$\therefore \quad \frac{x+2}{20} = \frac{x}{12} \quad \{\text{sides in the same ratio}\}$$

$$\therefore \quad 12(x+2) = 20x$$

$$\therefore \quad 12x + 24 = 20x$$

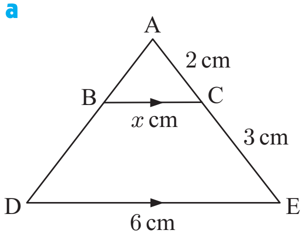
$$\therefore \quad 24 = 8x$$

$$\therefore \quad x = 3$$

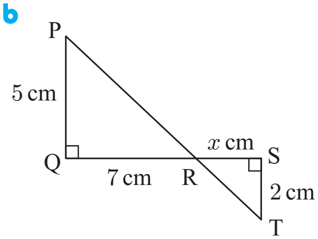
EXERCISE K.2

1 In each of the following, establish that a pair of triangles is similar, and hence find x:

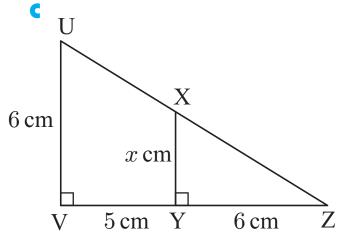
a



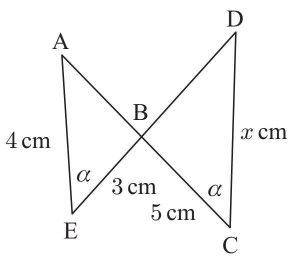
b



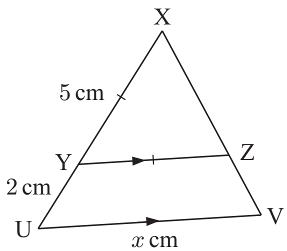
c



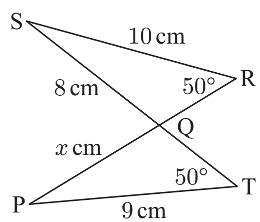
d



e



f



2 A father and son are standing side-by-side. The father is 1.8 m tall and casts a shadow 3.2 m long, while his son's shadow is 2.4 m long. How tall is the son?

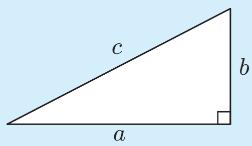
L PYTHAGORAS' THEOREM

The **hypotenuse** is the longest side of a right angled triangle. It is opposite the right angle.

Pythagoras' Theorem is:

In a right angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

$$c^2 = a^2 + b^2$$



This theorem, known to the ancient Greeks, is valuable because:

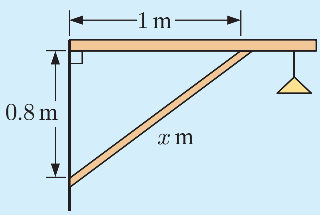
- if we know the lengths of any two sides of a right angled triangle then we can calculate the length of the third side
- if we know the lengths of the three sides then we can determine whether or not the triangle is right angled.

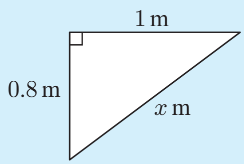
The second statement here relies on the **converse of Pythagoras' Theorem**, which is:

If a triangle has sides of length a, b and c units and $a^2 + b^2 = c^2$ then the triangle is right angled and its hypotenuse is c units long.

Example 25 **Self Tutor**

Find the unknown length in:





$x^2 = 0.8^2 + 1^2$

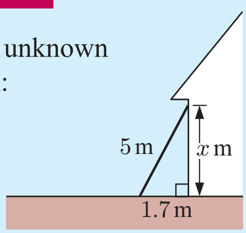
$\therefore \quad x = \sqrt{(0.8^2 + 1^2)}$

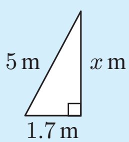
$\therefore \quad x \approx 1.2806$

So, the length is about 1.28 m.

Example 26 **Self Tutor**

Find the unknown length in:





$x^2 + 1.7^2 = 5^2$

$\therefore \quad x^2 = 5^2 - 1.7^2$

$\therefore \quad x = \sqrt{(5^2 - 1.7^2)}$

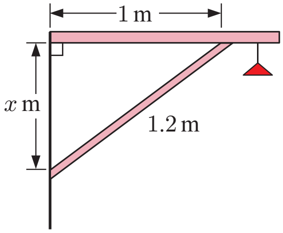
$\therefore \quad x \approx 4.7021$

So, the length is about 4.70 m.

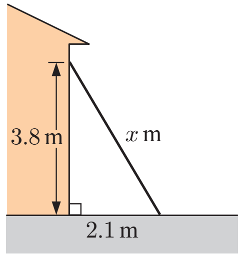
EXERCISE L.1

1 Find, correct to 3 significant figures, the value of x in:

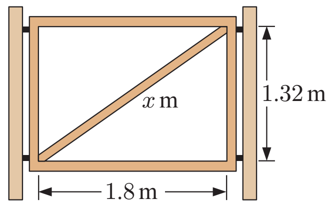
a



b

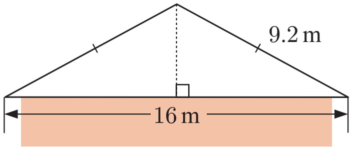


c

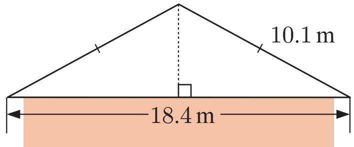


2 How high is the roof above the walls in the following roof structures?

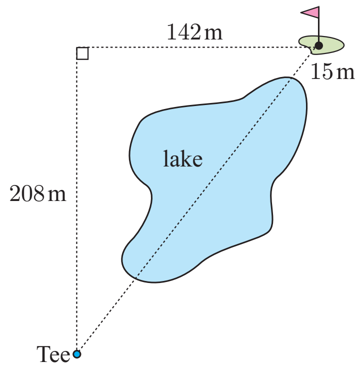
a



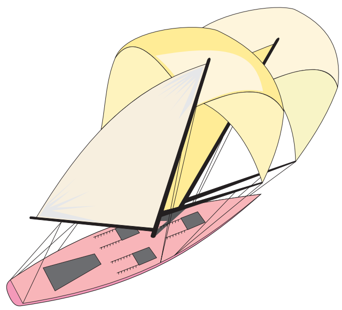
b



3 Bob is about to tee off on the sixth, a par 4 at the Royal Golf Club. He chooses to hit over the lake, directly at the flag. If the pin is 15 m from the water's edge, how far must he hit the ball to clear the lake?



4



A sailing ship sails 46 km north then 74 km east.

a Draw a fully labelled diagram of the ship's course.

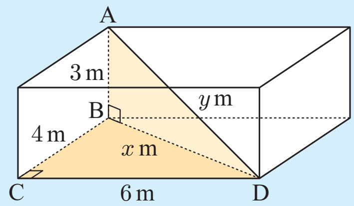
b How far is the ship from its starting point?

PYTHAGORAS' THEOREM IN 3-D PROBLEMS

The theorem of Pythagoras is often used twice in 3-D problem solving.

Example 27 **Self Tutor**

The floor of a room is 6 m by 4 m, and the floor to ceiling height is 3 m. Find the distance from a corner point on the floor to the opposite corner point on the ceiling.



The required distance is [AD]. We join [BD].

In ΔBCD, $x^2 = 4^2 + 6^2$ {Pythagoras}

$\therefore$ $x^2 = 16 + 36 = 52$

In ΔABD, $y^2 = x^2 + 3^2$

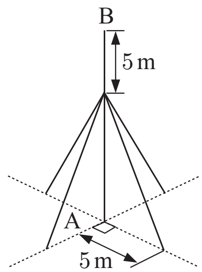
$\therefore$ $y^2 = 52 + 9 = 61$

$\therefore$ $y = \sqrt{61} \approx 7.81$

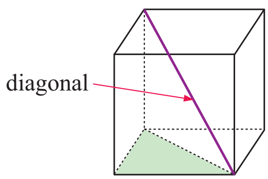
So, the distance is about 7.81 m.

EXERCISE L.2

1 A pole [AB] is 16 m tall. At a point 5 m below B, four wires are connected from the pole to the ground.
Each wire is pegged to the ground 5 m from the base of the pole.
What is the total length of wire needed if a total of 2 m extra is needed for tying?



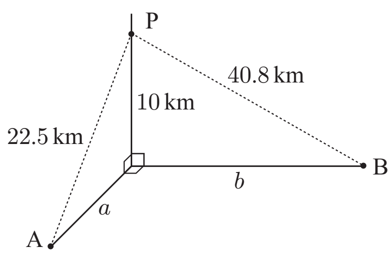
2 A cube has sides of length 10 cm.
Find the length of a diagonal of the cube.



3 A room is 7 m by 4 m and has a height of 3 m. Find the distance from a corner point on the floor to the opposite corner of the ceiling.

4 A pyramid of height 40 m has a square base with edges of length 50 m. Determine the length of the slant edges.

5



An aeroplane P is flying at an altitude of 10 000 m. The pilot spots two ships A and B. Ship A is due south of P and 22.5 km away in a direct line. Ship B is due east and 40.8 km from P in a direct line. Find the distance between the two ships.

M COORDINATE GEOMETRY

THE NUMBER PLANE

The position or location of any point in the **number plane** can be specified in terms of an **ordered pair** of numbers (x, y), where x is the **horizontal step** from a fixed point O, and y is the **vertical step** from O.

The point O is called the **origin**. Once O has been specified, we draw two perpendicular axes through it.

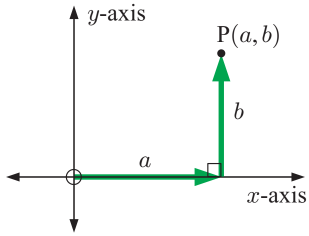
The **x-axis** is horizontal and the **y-axis** is vertical.

DEMO

The **number plane** is also known as either:

- the **2-dimensional plane**, or
- the **Cartesian plane**, named after **René Descartes**.

$(a,\ b)$ is called an **ordered pair**, where a and b are the coordinates of the point.
a is called the **x-coordinate**.
b is called the **y-coordinate**.



THE DISTANCE FORMULA

If $\text{A}(x_1,\ y_1)$ and $\text{B}(x_2,\ y_2)$ are two points in a plane, then the distance between these points is given by $\mathbf{AB} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Example 28 Self Tutor

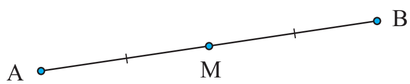
Find the distance between $\text{A}(-2,\ 1)$ and $\text{B}(3,\ 4)$.

$\text{A}(-2,\ 1)$ $\text{B}(3,\ 4)$
x_1 y_1 x_2 y_2

$$\begin{aligned}\text{AB} &= \sqrt{(3 - -2)^2 + (4-1)^2}\\ &= \sqrt{5^2 + 3^2}\\ &= \sqrt{25 + 9}\\ &= \sqrt{34} \text{ units}\end{aligned}$$

THE MIDPOINT FORMULA

If M is halfway between points A and B then M is the **midpoint** of [AB].



If $\text{A}(x_1,\ y_1)$ and $\text{B}(x_2,\ y_2)$ are two points then the **midpoint** M of [AB] has coordinates $\left(\dfrac{x_1 + x_2}{2},\ \dfrac{y_1 + y_2}{2}\right)$.

Example 29 Self Tutor

Find the coordinates of the midpoint of [AB] for $\text{A}(-1,\ 3)$ and $\text{B}(4,\ 7)$.

The x-coordinate of the midpoint $= \dfrac{-1+4}{2} = \frac{3}{2} = 1\frac{1}{2}$

The y-coordinate of the midpoint $= \dfrac{3+7}{2} = 5$

$\therefore$ the midpoint of [AB] is $(1\frac{1}{2},\ 5)$.

THE GRADIENT OR SLOPE OF A LINE

When looking at line segments drawn on a set of axes, it is clear that different line segments are inclined to the horizontal at different angles. Some appear to be steeper than others.

The **gradient** or **slope** of a line is a measure of its steepness.

If A is (x_1, y_1) and B is (x_2, y_2) then the **gradient** of [AB] is $\dfrac{y_2 - y_1}{x_2 - x_1}$.

Example 30 — Self Tutor

Find the gradient of the line through $(3, -2)$ and $(6, 4)$.

$(3, -2)$ $(6, 4)$

x_1 y_1 x_2 y_2

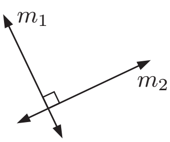
$$\text{gradient} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - -2}{6 - 3} = 2$$

PROPERTIES OF GRADIENT

- horizontal lines have a gradient of **0 (zero)**
- vertical lines have an **undefined** gradient
-  forward sloping lines have **positive** gradients
-  backward sloping lines have **negative** gradients
- **parallel lines** have equal gradients
- the gradients of two **perpendicular lines** are *negative reciprocals* of each other.

  If the gradients are m_1 and m_2 then $\boldsymbol{m_2 = \dfrac{-1}{m_1}}$ or $\boldsymbol{m_1 m_2 = -1}$.

  This is true except when the lines are parallel to the axes.



EQUATIONS OF LINES

The **equation of a line** states the connection between the x and y values for every point on the line, and only for points on the line

Equations of lines have various forms:

- All **vertical lines** have equations of the form $x = a$ where a is a constant.
- All **horizontal lines** have equations of the form $y = c$ where c is a constant.
- If a straight line has gradient m and passes through (a, b)

  then it has equation $\boldsymbol{\dfrac{y - b}{x - a} = m}$ or $\boldsymbol{y - b = m(x - a)}$ {**point-gradient** form}

  which can be rearranged into $\boldsymbol{y = mx + c}$ {**gradient-intercept** form}

- If a straight line has gradient $\frac{A}{B}$ and passes through (x_1, x_2) then it has equation $\boldsymbol{Ax - By = Ax_1 - By_1}$ or $\boldsymbol{Ax - By = C}$ {**general** form}

Example 31 Self Tutor

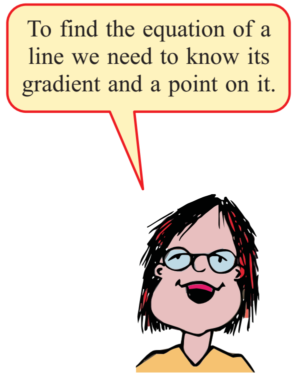
Find, in gradient-intercept form, the equation of the line through $(-1, 3)$ with a gradient of 5.

The equation of the line is $\frac{y-3}{x--1} = 5$

$\therefore \frac{y-3}{x+1} = 5$

$\therefore y - 3 = 5(x+1)$

$\therefore y = 5x + 8$



Example 32 Self Tutor

Find, in general form, the equation of the line through $(1, -5)$ and $(5, -2)$.

The gradient $= \frac{-2 - -5}{5-1}$

$= \frac{3}{4}$

So, the equation is $\frac{y - -2}{x-5} = \frac{3}{4}$

$\therefore \frac{y+2}{x-5} = \frac{3}{4}$

$\therefore 4y + 8 = 3x - 15$

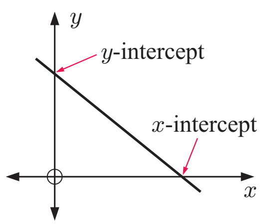
$\therefore 3x - 4y = 23$

AXES INTERCEPTS

Axes intercepts are the x- and y-values where a graph cuts the coordinate axes.

The x-intercept is found by letting $y = 0$.

The y-intercept is found by letting $x = 0$.



Example 33 Self Tutor

For the line with equation $2x - 3y = 12$, find the axes intercepts.

When $x = 0$, $-3y = 12$

$\therefore y = -4$

When $y = 0$, $2x = 12$

$\therefore x = 6$

So, the y-intercept is -4 and the x-intercept is 6.

DOES A POINT LIE ON A LINE?

A point lies on a line if its coordinates satisfy the equation of the line.

Example 34 — Self Tutor

Does $(3,\ -2)$ lie on the line with equation $5x - 2y = 20$?

Substituting $(3,\ -2)$ into $5x - 2y = 20$ gives

$$\begin{aligned} \text{LHS} &= 5(3) - 2(-2) = 19 \\ \therefore\ \text{LHS} &\neq \text{RHS} \end{aligned}$$

$\therefore$ $(3,\ -2)$ does not lie on the line.

FINDING WHERE GRAPHS MEET

Example 35 — Self Tutor

Use graphical methods to find where the lines $x + y = 6$ and $2x - y = 6$ meet.

For $x + y = 6$:

when $x = 0,\ y = 6$
when $y = 0,\ x = 6$

x	0	6
y	6	0

For $2x - y = 6$:

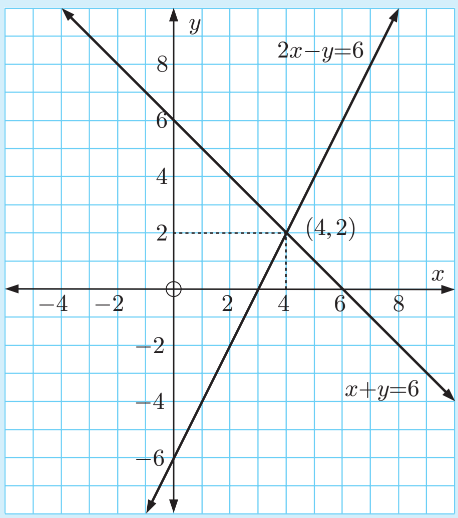
when $x = 0,\ -y = 6$
$\therefore\ y = -6$

when $y = 0,\ 2x = 6$
$\therefore\ x = 3$

x	0	3
y	-6	0

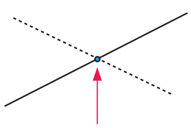
The graphs meet at $(4,\ 2)$.

Check: $4 + 2 = 6$ ✓
$2 \times 4 - 2 = 6$ ✓



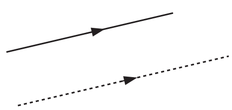
When determining whether two lines meet, there are three possible situations which may occur. These are:

Case 1:



The lines meet in a single **point of intersection**.

Case 2:



The lines are **parallel** and **never meet**. There is no point of intersection.

Case 3:

The lines are **coincident**. There are infinitely many points of intersection.

EXERCISE M

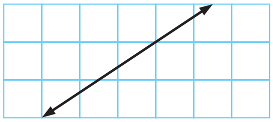
1 Use the distance formula to find the distance between the following pairs of points:

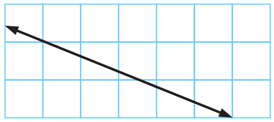
a A(1, 3) and B(4, 5)
b O(0, 0) and C(3, −5)
c P(5, 2) and Q(1, 4)
d S(0, −3) and T(−1, 0).

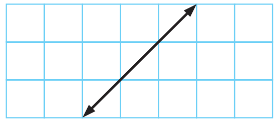
2 Find the midpoint of [AB] for:

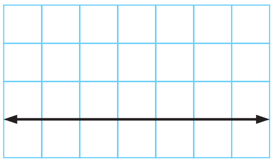
a A(3, 6) and B(1, 0)
b A(5, 2) and B(−1, −4)
c A(7, 0) and B(0, 3)
d A(5, −2) and B(−1, −3).

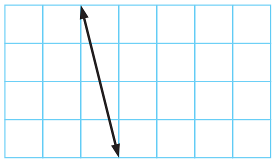
3 By finding a y-step and an x-step, determine the gradient of each of the following lines:

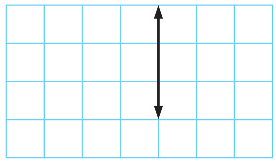
a


b


c


d


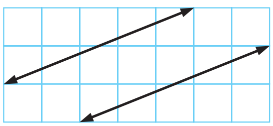
e


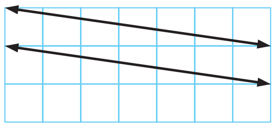
f


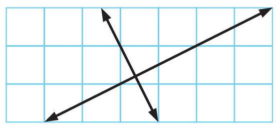
4 Find the gradient of the line passing through:

a (2, 3) and (4, 7)
b (3, 2) and (5, 8)
c (−1, 2) and (−1, 5)
d (4, −3) and (−1, −3)
e (0, 0) and (−1, 4)
f (3, −1) and (−1, −2).

5 Classify the following pairs of lines as parallel, perpendicular, or neither. Give reasons for your answers.

a


b


c


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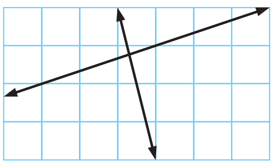
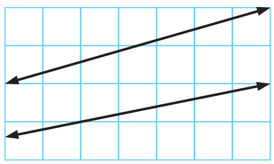


e


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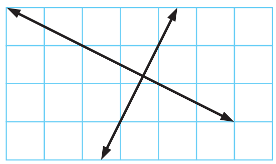


6 State the gradient of the line which is perpendicular to the line with gradient:

a $\frac{3}{4}$ **b** $\frac{11}{3}$ **c** 4 **d** $-\frac{1}{3}$ **e** −5 **f** 0

7 Find, in gradient-intercept form, the equation of the line through:

a (4, 1) with gradient 2
b (1, 2) with gradient −2
c (5, 0) with gradient 3
d (−1, 7) with gradient −3
e (1, 5) with gradient −4
f (2, 7) with gradient 1.

8 Find, in general form, the equation of the line through:

a $(2, 1)$ with gradient $\frac{3}{2}$
b $(1, 4)$ with gradient $-\frac{3}{2}$
c $(4, 0)$ with gradient $\frac{1}{3}$
d $(0, 6)$ with gradient -4
e $(-1, -3)$ with gradient 3
f $(4, -2)$ with gradient $-\frac{4}{9}$.

9 Find the equations of the lines through:

a $(0, 1)$ and $(3, 2)$
b $(1, 4)$ and $(0, -1)$
c $(2, -1)$ and $(-1, -4)$
d $(0, -2)$ and $(5, 2)$
e $(3, 2)$ and $(-1, 0)$
f $(-1, -1)$ and $(2, -3)$

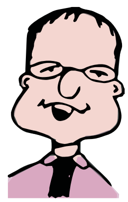
10 Find the equations of the lines through:

a $(3, -2)$ and $(5, -2)$
b $(6, 7)$ and $(6, -11)$
c $(-3, 1)$ and $(-3, -3)$

11 Copy and complete:

	Equation of line	*Gradient*	*x-intercept*	*y-intercept*
a	$2x - 3y = 6$			
b	$4x + 5y = 20$			
c	$y = -2x + 5$			
d	$x = 8$			
e	$y = 5$			
f	$x + y = 11$			
g	$4x + y = 8$			
h	$x - 3y = 12$			

If a line has equation $y = mx + c$ then the gradient of the line is m and the y-intercept is c.



12 **a** Does $(3, 4)$ lie on the line with equation $3x - 2y = 1$?
b Does $(-2, 5)$ lie on the line with equation $5x + 3y = -5$?
c Does $(6, -\frac{1}{2})$ lie on the line $3x - 8y = 22$?

13 Use graphical methods to find where the following lines meet:

a $x + 2y = 8$, $y = 2x - 6$
b $y = -3x - 3$, $3x - 2y = -12$
c $3x + y = -3$, $2x - 3y = -24$
d $2x - 3y = 8$, $3x + 2y = 12$
e $x + 3y = 10$, $2x + 6y = 11$
f $5x + 3y = 10$, $10x + 6y = 20$

GRAPHING PACKAGE

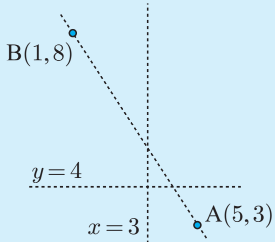
Example 36 | **Self Tutor**

A straight road is to pass through the points A$(5, 3)$ and B$(1, 8)$.

a Find where this road meets the road given by:
  i $x = 3$ **ii** $y = 4$

b If we wish to refer to the points on the road (AB) *between* A and B, how can we indicate this?

c Does C$(23, -20)$ lie on the road (AB)?



The line representing the road has gradient $m = \dfrac{3-8}{5-1} = -\frac{5}{4}$.

So, its equation is $\dfrac{y-3}{x-5} = -\dfrac{5}{4}$

$\therefore \quad 4(y-3) = -5(x-5)$

$\therefore \quad 4y - 12 = -5x + 25$

$\therefore \quad 5x + 4y = 37$

a **i** When $x = 3$, $5(3) + 4y = 37$
$\therefore \quad 15 + 4y = 37$
$\therefore \quad 4y = 22$
$\therefore \quad y = 5\frac{1}{2}$
$\therefore$ they meet at $(3, 5\frac{1}{2})$.

ii When $y = 4$, $5x + 4(4) = 37$
$\therefore \quad 5x + 16 = 37$
$\therefore \quad 5x = 21$
$\therefore \quad x = \frac{21}{5}$
$\therefore$ they meet at $(\frac{21}{5}, 4)$.

b We restrict the possible x-values to $1 \leqslant x \leqslant 5$.

c If C$(23, -20)$ lies on the line, its coordinates must satisfy the line's equation.

Now LHS $= 5(23) + 4(-20)$
$= 115 - 80$
$= 35 \neq 37$ $\quad \therefore$ C does not lie on the road.

14 Find the equation of the:

a horizontal line through $(3, -4)$

b vertical line with x-intercept 5

c vertical line through $(-1, -3)$

d horizontal line with y-intercept 2

e x-axis

f y-axis.

15 Find the equation of the line:

a through A$(-1, 4)$ which has gradient $\frac{3}{4}$

b through P$(2, -5)$ and Q$(7, 0)$

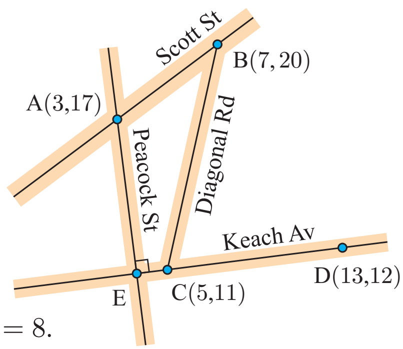
c parallel to the line with equation $y = 3x - 2$ and passing through $(0, 0)$

d parallel to the line with equation $2x + 3y = 8$ and passing through $(-1, 7)$

e perpendicular to the line with equation $y = -2x + 5$ and passing through $(3, -1)$

f perpendicular to the line with equation $3x - y = 11$ and passing through $(-2, 5)$.

16 A is the town hall on Scott Street and D is a Post Office on Keach Avenue. Diagonal Road intersects Scott Street at B and Keach Avenue at C.



a Find the equation of Keach Avenue.

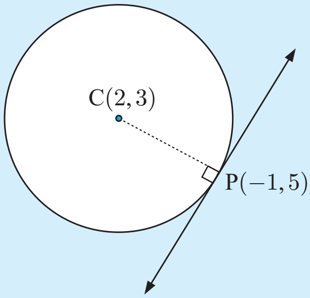
b Find the equation of Peacock Street.

c Find the equation of Diagonal Road. (Be careful!)

d Plunkit Street lies on the map reference line $x = 8$. Where does Plunkit Street intersect Keach Avenue?

Example 37 **Self Tutor**

Find the equation of the tangent to the circle with centre $(2,\ 3)$ at the point $(-1,\ 5)$.



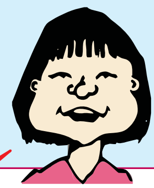
The gradient of [CP] is $\dfrac{3-5}{2-(-1)} = \dfrac{-2}{3} = -\frac{2}{3}$

$\therefore$ the gradient of the tangent at P is $\frac{3}{2}$.

Since the tangent passes through $(-1,\ 5)$,

its equation is $\dfrac{y-5}{x--1} = \dfrac{3}{2}$

$$\therefore\ 2(y-5) = 3(x+1)$$
$$\therefore\ 2y - 10 = 3x + 3$$
$$\therefore\ 3x - 2y = -13$$



17 Find the equation of the tangent to the circle with centre:

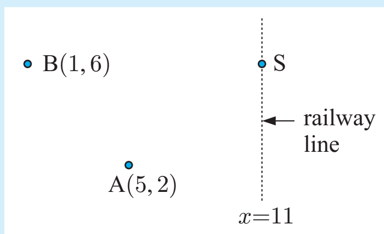
a $(0,\ 2)$ at the point $(-1,\ 5)$

b $(3,\ -1)$ at the point $(-1,\ 1)$

c $(2,\ -2)$ at the point $(5,\ -2)$.

Example 38 **Self Tutor**

Mining towns are situated at B$(1,\ 6)$ and A$(5,\ 2)$. Where should the railway siding S be located so that ore trucks from either A or B would travel equal distances to a railway line with equation $x = 11$?

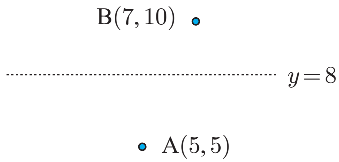


Suppose S has the coordinates $(11,\ a)$.

$$\text{Now}\quad \text{BS} = \text{AS}$$
$$\therefore\ \sqrt{(11-1)^2 + (a-6)^2} = \sqrt{(11-5)^2 + (a-2)^2}$$
$$\therefore\ 10^2 + (a-6)^2 = 6^2 + (a-2)^2 \qquad \{\text{squaring both sides}\}$$
$$\therefore\ 100 + a^2 - 12a + 36 = 36 + a^2 - 4a + 4$$
$$\therefore\ -12a + 4a = 4 - 100$$
$$\therefore\ -8a = -96$$
$$\therefore\ a = 12$$

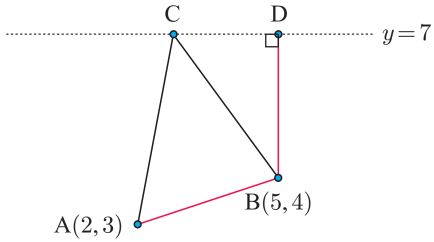
So, the railway siding should be located at $(11,\ 12)$.

18 A$(5,\ 5)$ and B$(7,\ 10)$ are houses and $y = 8$ is a gas pipeline. Where should the one outlet from the pipeline be placed so that it is the same distance from both houses?

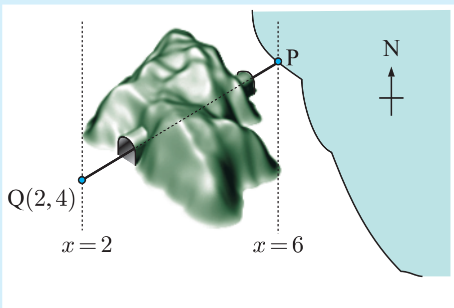


19 (CD) is a water pipeline. A and B are two towns. A pumping station is to be located on the pipeline to pump water to A and B. Each town is to pay for their own service pipes and they insist on equality of costs.

- **a** Where should C be located to ensure equality of costs?
- **b** What is the total length of service pipe required?
- **c** If the towns agree to pay equal amounts, would it be cheaper to install the service pipeline from D to B to A?



Example 39 **Self Tutor**



Scale: each grid unit is 1 km.

A tunnel through the mountains connects town Q$(2,\ 4)$ to the port at P.

P is on grid reference $x = 6$ and the distance between the town and the port is 5 km.

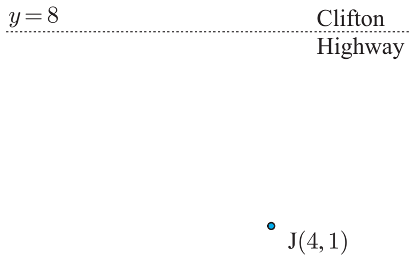
Assuming the diagram is reasonably accurate, find the horizontal grid reference of the port.

Suppose P is at $(6,\ a)$.

$$\begin{aligned}
\text{Now} \quad PQ &= 5 \\
\therefore \quad \sqrt{(6-2)^2 + (a-4)^2} &= 5 \\
\therefore \quad \sqrt{16 + (a-4)^2} &= 5 \\
\therefore \quad 16 + (a-4)^2 &= 25 \\
\therefore \quad (a-4)^2 &= 9 \\
\therefore \quad a - 4 &= \pm 3 \\
\therefore \quad a &= 4 \pm 3 = 7 \text{ or } 1
\end{aligned}$$

From the diagram, P is further north than Q, and so $a > 4$

So, P is at $(6,\ 7)$ and the horizontal grid reference is $y = 7$.

20 

Scale: 1 unit $\equiv$ 1 km.

Jason's girlfriend lives in a house on Clifton Highway which has equation $y = 8$. The distance 'as the crow flies' from Jason's house to his girlfriend's house is 11.73 km. If Jason lives at $(4, 1)$, what are the coordinates of his girlfriend's house?

21 **a** A circle has centre (a, b) and radius r units. $P(x, y)$ moves on the circle. Show that $(x-a)^2 + (y-b)^2 = r^2$.

b Find the equation of the circle with:

i centre $(4, 3)$ and radius 5 units **ii** centre $(-1, 5)$ and radius 2 units

iii centre $(0, 0)$ and radius 10 units

iv ends of a diameter $(-1, 5)$ and $(3, 1)$.

22 Find the centre and radius of the circle:

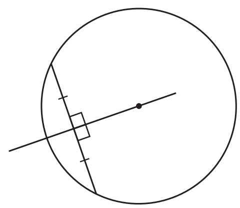
a $(x-1)^2 + (y-3)^2 = 4$ **b** $x^2 + (y+2)^2 = 16$ **c** $x^2 + y^2 = 7$

23 Consider the circle with equation $(x-2)^2 + (y+3)^2 = 20$.

a State the circle's centre and radius. **b** Show that $(4, 1)$ lies on the circle.

c Find the equation of the tangent to the circle at the point $(4, 1)$.

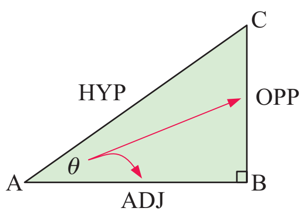
24 The perpendicular bisector of a chord of a circle passes through the centre of the circle.
Find the centre of a circle passing through points $P(5, 7)$, $Q(7, 1)$ and $R(-1, 5)$ by finding the perpendicular bisectors of [PQ] and [QR] and solving them simultaneously.



N RIGHT ANGLED TRIANGLE TRIGONOMETRY

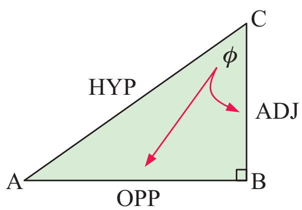
LABELLING RIGHT ANGLED TRIANGLES

The **hypotenuse (HYP)** is the side which is opposite the right angle. It is the longest side of the triangle.



For the angle marked θ:

- [BC] is the side **opposite (OPP)** angle θ
- [AB] is the side **adjacent (ADJ)** angle θ.

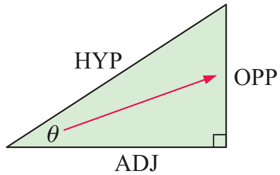


For the angle marked ϕ:

- [AB] is the side **opposite (OPP)** angle ϕ
- [BC] is the side **adjacent (ADJ)** angle ϕ.

THE THREE BASIC TRIGONOMETRIC RATIOS

$$\sin\theta = \frac{\text{OPP}}{\text{HYP}}, \quad \cos\theta = \frac{\text{ADJ}}{\text{HYP}}, \quad \tan\theta = \frac{\text{OPP}}{\text{ADJ}}$$

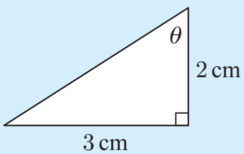


$\sin\theta$, $\cos\theta$ and $\tan\theta$ are abbreviations for sine θ, cosine θ and tangent θ.

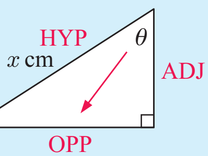
The three formulae above are called the **trigonometric ratios** and are the tools we use for finding side lengths and angles of right angled triangles.

However, before doing this we will calculate the trigonometric ratios in right angled triangles where we know two of the sides.

Example 40 Self Tutor



Find, without using a calculator, $\sin\theta$, $\cos\theta$ and $\tan\theta$.



If the hypotenuse is x cm long

$$x^2 = 2^2 + 3^2 \quad \{\text{Pythagoras}\}$$
$$\therefore \quad x^2 = 13$$
$$\therefore \quad x = \sqrt{13} \quad \{\text{as } x > 0\}$$

So, $\sin\theta = \dfrac{\text{OPP}}{\text{HYP}} = \frac{3}{\sqrt{13}}$, $\cos\theta = \dfrac{\text{ADJ}}{\text{HYP}} = \frac{2}{\sqrt{13}}$, $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \frac{3}{2}$.

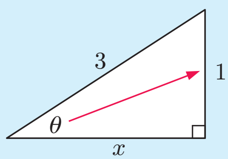
Example 41 Self Tutor

If θ is an acute angle and $\sin\theta = \frac{1}{3}$, find $\cos\theta$ and $\tan\theta$ without using a calculator.

We draw a right angled triangle and mark on angle θ so that OPP $= 1$ unit and HYP $= 3$ units.

Now $x^2 + 1^2 = 3^2$ {Pythagoras}

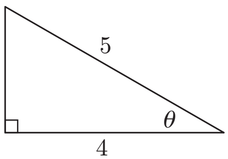
$$\therefore \quad x^2 + 1 = 9$$
$$\therefore \quad x^2 = 8$$
$$\therefore \quad x = \sqrt{8} \quad \{\text{as } x > 0\}$$

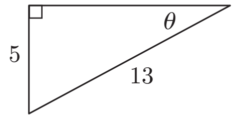


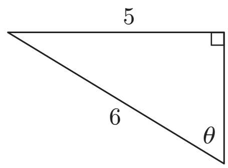
$\therefore \quad \cos\theta = \dfrac{\text{ADJ}}{\text{HYP}} = \frac{\sqrt{8}}{3}$ and $\tan\theta = \dfrac{\text{OPP}}{\text{ADJ}} = \frac{1}{\sqrt{8}}$.

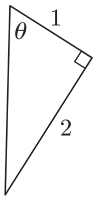
EXERCISE N.1

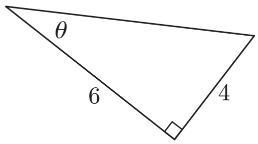
1 For the following triangles, find the length of the third side and hence find $\sin\theta$, $\cos\theta$ and $\tan\theta$:

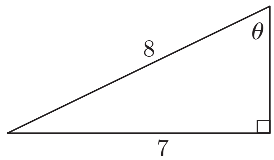
a


b


c


d


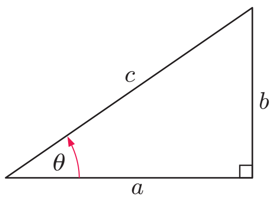
e


f


2 **a** If θ is an acute angle and $\cos\theta = \frac{1}{2}$, find $\sin\theta$ and $\tan\theta$.

b If α is an acute angle and $\sin\alpha = \frac{2}{3}$, find $\cos\alpha$ and $\tan\alpha$.

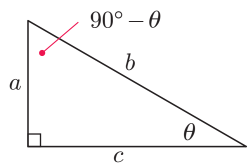
c If β is an acute angle and $\tan\beta = \frac{4}{3}$, find $\sin\beta$ and $\cos\beta$.

3


a For the triangle given, write down expressions for $\sin\theta$, $\cos\theta$ and $\tan\theta$.

b Write $\dfrac{\sin\theta}{\cos\theta}$ in terms of a, b and c and hence show that $\dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

4 The angle marked $90^o - \theta$ is the **complement** of θ.



a Find:

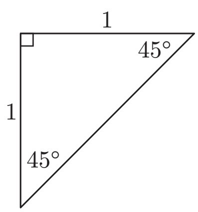
i $\sin\theta$ **ii** $\cos\theta$

iii $\sin(90^o - \theta)$ **iv** $\cos(90^o - \theta)$

b Use your results of **a** to complete the following statements:

i *The sine of an angle is the cosine of its ......*

ii *The cosine of an angle is the sine of its ......*

5


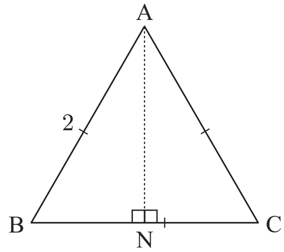
a Find the length of the remaining side.

b Find $\sin 45^o$, $\cos 45^o$, $\tan 45^o$ using the figure.

c Use your calculator to check your answers.

6 Triangle ABC is equilateral. [AN] is the altitude corresponding to side [BC].

- **a** State the measures of $\widehat{ABN}$ and $\widehat{BAN}$.
- **b** Find the lengths of [BN] and [AN].
- **c** Without using a calculator, find:
  - **i** $\sin 60^o$, $\cos 60^o$ and $\tan 60^o$
  - **ii** $\sin 30^o$, $\cos 30^o$ and $\tan 30^o$.



COMMON TRIGONOMETRIC RATIOS

We can summarise the ratios for special angles in table form. Try to learn them.

θ	0^o	30^o	45^o	60^o	90^o
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

FINDING SIDES AND ANGLES

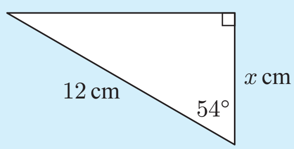
Before commencing calculations, check that the MODE on your calculator is set on **degrees**. In this chapter all angle measure is in degrees.

In a right angled triangle, if we wish to find the **length of a side**, we first need to know one angle and one other side.

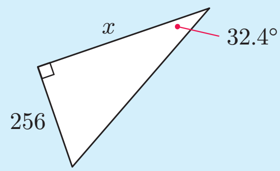
Example 42 Self Tutor

Find, correct to 3 significant figures, the value of x in:

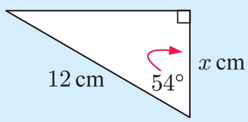
a



b



a



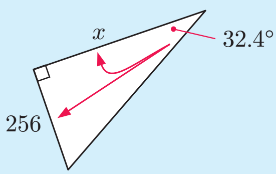
For the 54^o angle, HYP $= 12$, ADJ $= x$.

$\therefore \quad \cos 54^o = \dfrac{x}{12}$

$\therefore \quad 12\cos 54^o = x$

$\therefore \quad x \approx 7.05$

b



For the 32.4^o angle, OPP $= 12$, ADJ $= x$.

So, $\tan 32.4^o = \dfrac{256}{x}$

$\therefore \quad x = \dfrac{256}{\tan 32.4^o} \approx 403.4$

$\therefore \quad x \approx 403$

In a right angled triangle, if we wish to find the **size of an acute angle** we need to know the lengths of two sides. We then need to find the appropriate **inverse** trigonometric ratio:

- If $\sin\theta = \frac{a}{b}$ then $\theta = \sin^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a sine of $\frac{a}{b}$'.
- If $\cos\theta = \frac{a}{b}$ then $\theta = \cos^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a cosine of $\frac{a}{b}$'.
- If $\tan\theta = \frac{a}{b}$ then $\theta = \tan^{-1}\left(\frac{a}{b}\right)$ which reads 'the angle with a tangent of $\frac{a}{b}$'.

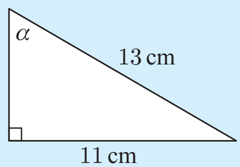
An alternative notation for the three inverse trigonometric functions is:

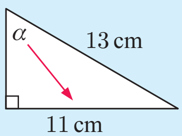
- $\arcsin\theta$ for $\sin^{-1}\theta$
- $\arccos\theta$ for $\cos^{-1}\theta$
- $\arctan\theta$ for $\tan^{-1}\theta$

Find help using your calculator to find inverse trigonometric ratios, consult the graphics calculator instructions chapter.

Example 43 Self Tutor

Find α in degrees, correct to 3 significant figures:





For angle α, OPP $= 11$, HYP $= 13$.

$\therefore \quad \sin\alpha = \frac{11}{13}$

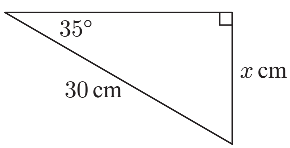
$\therefore \quad \alpha = \sin^{-1}\left(\frac{11}{13}\right)$

$\therefore \quad \alpha \approx 57.8^o$

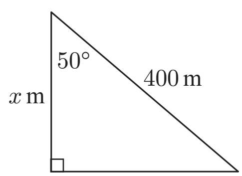
EXERCISE N.2

1 Find, correct to 3 significant figures, the value of the unknown in each of the following:

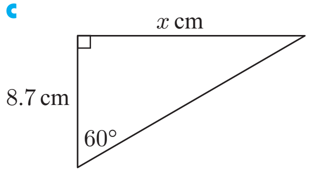
a



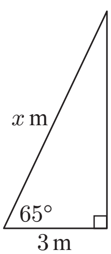
b



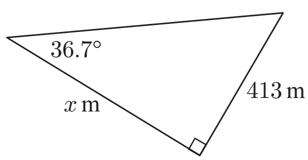
c



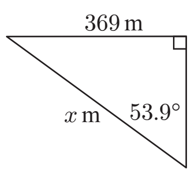
d



e



f



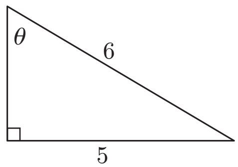
2 Use your calculator to find the acute angle θ, to 3 significant figures, if:

a $\sin\theta = 0.9364$ **b** $\cos\theta = 0.2381$ **c** $\tan\theta = 1.7321$ **d** $\cos\theta = \frac{2}{7}$

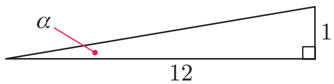
e $\sin\theta = \frac{1}{3}$ **f** $\tan\theta = \frac{14}{3}$ **g** $\sin\theta = \frac{\sqrt{3}}{11}$ **h** $\cos\theta = \frac{5}{\sqrt{37}}$

3 Find, correct to 3 significant figures, the measure of the unknown angle in each of the following:

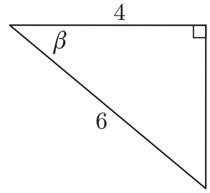
a



b

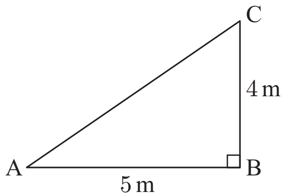


c

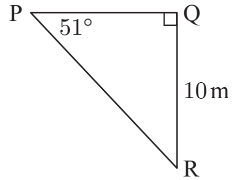


4 Find all unknown side lengths and angles of the following triangles:

a

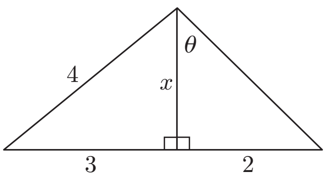


b

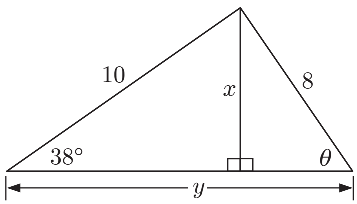


5 Find all unknown sides and angles in:

a



b



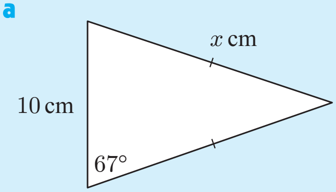
ISOSCELES TRIANGLES

To use trigonometry with isosceles triangles we invariably draw the **perpendicular** from the apex to the base. This altitude **bisects** the base.

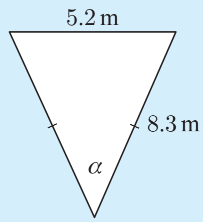
Example 44 **Self Tutor**

Find the unknowns in the following diagrams:

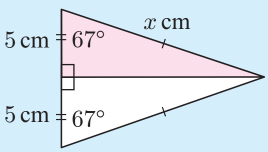
a



b



a

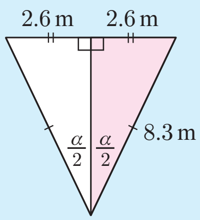


In the shaded right angled triangle,

$$\cos 67^o = \frac{5}{x}$$

$$\therefore \quad x = \frac{5}{\cos 67^o} \approx 12.8$$

b



In the shaded right angled triangle,

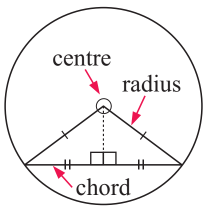
$$\sin\left(\frac{\alpha}{2}\right) = \frac{2.6}{8.3}$$

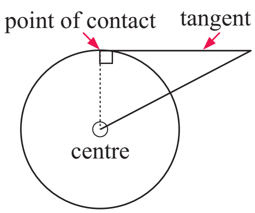
$$\therefore \quad \frac{\alpha}{2} = \sin^{-1}\left(\frac{2.6}{8.3}\right)$$

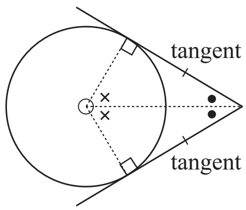
$$\therefore \quad \alpha = 2\sin^{-1}\left(\frac{2.6}{8.3}\right) \approx 36.5^o$$

CHORDS AND TANGENTS

Right angled triangles occur in chord and tangent problems.

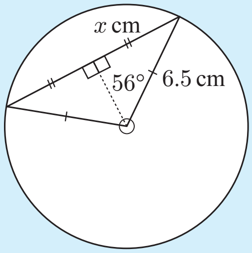






Example 45 **Self Tutor**

A chord of a circle subtends an angle of 112^o at its centre. Find the length of the chord if the radius of the circle is 6.5 cm.



We complete an isosceles triangle and draw the line from the apex to the base.

For the 56^o angle, HYP $= 6.5$, OPP $= x$

$$\therefore \quad \sin 56^o = \frac{x}{6.5}$$

$$\therefore \quad 6.5 \times \sin 56^o = x$$

$$\therefore \quad x \approx 5.389$$

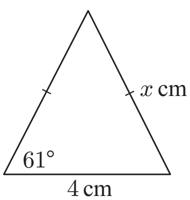
$$\therefore \quad 2x \approx 10.78$$

$\therefore$ the chord is about 10.8 cm long.

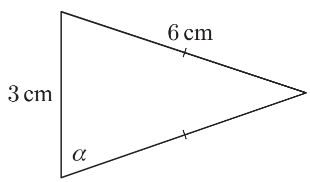
EXERCISE N.3

1 Find, correct to 4 significant figures, the unknowns in the following:

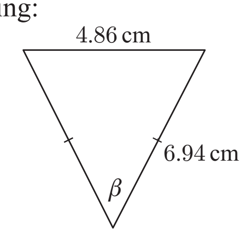
a



b

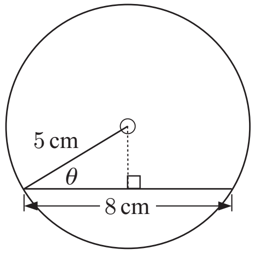


c

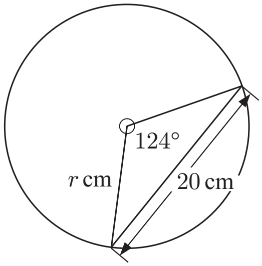


2 Find the value of the unknown in:

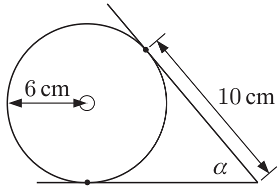
a



b



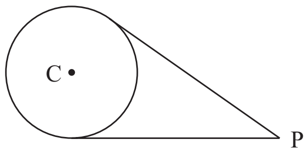
c



3 A chord of a circle subtends an angle of 89^o at its centre. Find the length of the chord given that the circle's diameter is 11.4 cm.

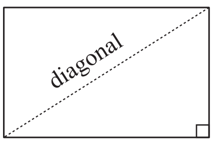
4 A chord of a circle is 13.2 cm long and the circle's radius is 9.4 cm. Find the angle subtended by the chord at the centre of the circle.

5 Point P is 10 cm from the centre of a circle of radius 4 cm. Tangents are drawn from P to the circle. Find the angle between the tangents.

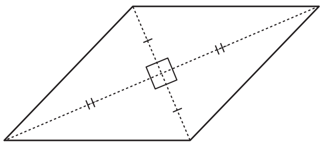


OTHER FIGURES

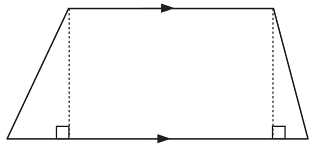
Right angled triangles can also be found in other geometric figures such as rectangles, rhombi, and trapezia.



rectangle



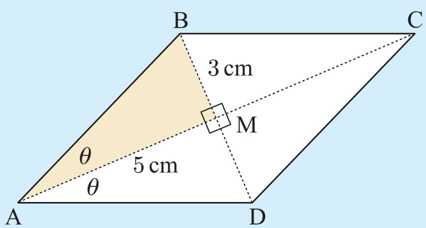
rhombus



trapezium or trapezoid

Example 46 **Self Tutor**

A rhombus has diagonals of length 10 cm and 6 cm respectively.
Find the smaller angle of the rhombus.



The diagonals bisect each other at right angles, so AM = 5 cm and BM = 3 cm.

In ΔABM, θ will be the smallest angle as it is opposite the shortest side.

$$\tan\theta = \tfrac{3}{5}$$

$$\therefore \quad \theta = \tan^{-1}(\tfrac{3}{5})$$

$$\therefore \quad \theta \approx 30.964^o$$

The required angle is 2θ as the diagonals bisect the angles at each vertex.
So, the angle is about 61.9^o.

EXERCISE N.4

1 A rectangle is 9.2 m by 3.8 m. What angle does its diagonal make with its longer side?

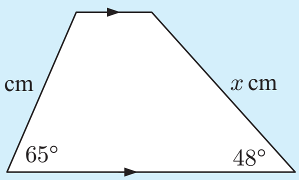
2 The diagonal and the longer side of a rectangle make an angle of 43.2^o. If the longer side is 12.6 cm, find the length of the shorter side.

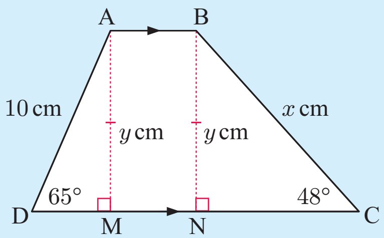
3 A rhombus has diagonals of length 12 cm and 7 cm respectively. Find the larger angle of the rhombus.

4 The smaller angle of a rhombus measures 21.8^o and the shorter diagonal has length 13.8 cm. Find the lengths of the sides of the rhombus.

Example 47 **Self Tutor**

Find x given:





We draw perpendiculars [AM] and [BN] to [DC], creating right angled triangles and the rectangle ABNM.

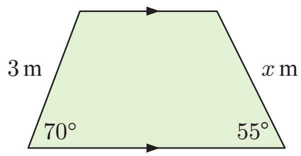
In ΔADM, $\sin 65^o = \dfrac{y}{10}$

$\therefore \quad y = 10 \sin 65^o$

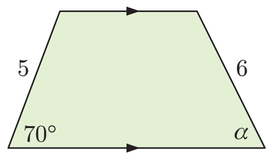
In ΔBCN, $\sin 48^o = \dfrac{y}{x} = \dfrac{10 \sin 65^o}{x}$

$\therefore \quad x = \dfrac{10 \sin 65^o}{\sin 48^o} \approx 12.2$

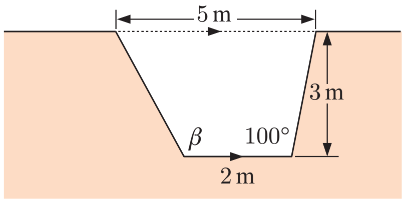
5 **a** Find the value of x in:



b Find the unknown angle in:

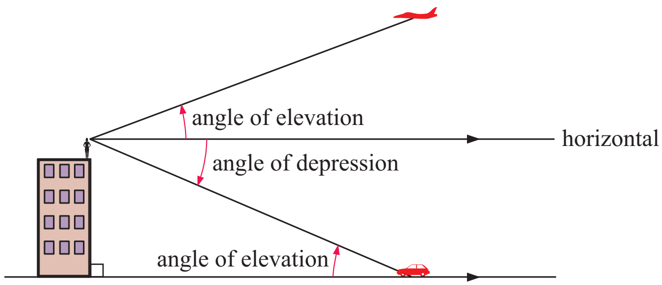


6 A stormwater drain is to have the shape illustrated. Determine the angle β the left hand side makes with the bottom of the drain.



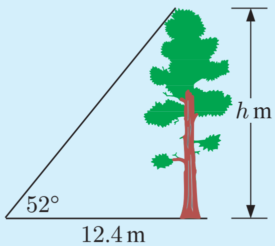
PROBLEM SOLVING USING TRIGONOMETRY

Trigonometry is a very useful branch of mathematics. **Heights** and **distances** which are very difficult or even impossible to measure can often be found using **trigonometry**.



Example 48 Self Tutor

Find the height of a tree which casts a shadow of 12.4 m when the sun makes an angle of 52^o to the horizon.



Let h m be the tree's height.

For the 52^o angle, OPP $= h$, ADJ $= 12.4$

$$\therefore \quad \tan 52^o = \frac{h}{12.4}$$

$$\therefore \quad 12.4 \times \tan 52^o = h$$

$$\therefore \quad h \approx 15.9$$

$\therefore$ the tree is 15.9 m high.

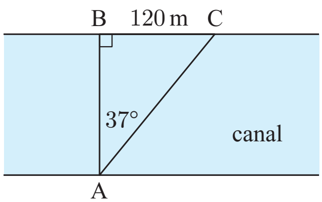
EXERCISE N.5

1 Find the height of a flagpole which casts a shadow of 9.32 m when the sun makes an angle of 63^o to the horizontal.

2 A hill is inclined at 18^o to the horizontal. It runs down to the beach so its base is at sea level.

- **a** If I walk 1.2 km up the hill, what is my height above sea level?
- **b** If I am 500 metres above sea level, how far have I walked up the hill?

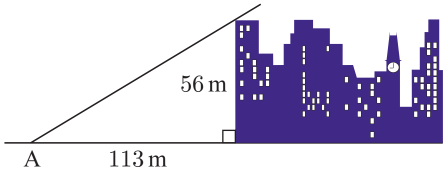
3



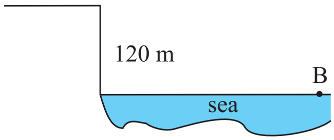
A surveyor standing at A notices two posts B and C on the opposite side of a canal. The posts are 120 m apart. If the angle of sight between the posts is 37^o, how wide is the canal?

4 A train must climb a constant gradient of 5.5 m for every 200 m of track. Find the angle of incline.

5 Find the angle of elevation to the top of a 56 m high building from point A which is 113 m from its base. What is the angle of depression from the top of the building to A?



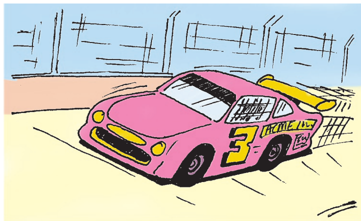
6 The angle of depression from the top of a 120 m high vertical cliff to a boat B is 16^o.
Find how far the boat is from the base of the cliff.



7 Sarah measures the angle of elevation to the top of a tree as 23.6^o from a point which is 250 m from its base. Her eye level, from which the angle measurement is taken, is 1.5 m above the ground. Assuming the ground is horizontal, find the height of the tree.

8 Kylie measures the angle of elevation from a point on level ground to the top of a building 120 metres high to be 32^o. She walks towards the building until the angle of elevation is 45^o. How far did she walk?

9 A circular track of radius r m is banked at an angle of θ to the horizontal. The ideal speed for the bend is given by the formula $s = \sqrt{gr\tan\theta}$ where $g = 9.8\ \text{m s}^{-2}$.

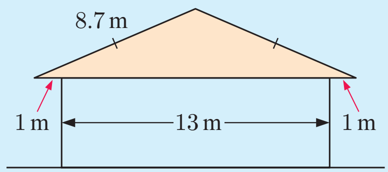


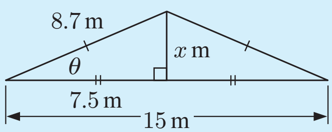
a What is the ideal speed for a vehicle travelling on a circular track of radius 100 m which is banked at an angle of 15^o?

b At what angle should a track of radius 200 m be banked if it is designed for a vehicle travelling at $20\ \text{m s}^{-1}$?

Example 49 **Self Tutor**

A builder has designed the roof structure illustrated. The pitch of the roof is the angle that the roof makes with the horizontal. Find the pitch of this roof.





By constructing an altitude of the isosceles triangle, we form two right angled triangles.
For angle θ:

ADJ = 7.5, HYP = 8.7

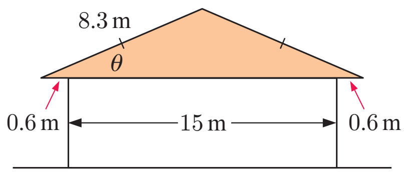
$$\therefore\quad \cos\theta = \frac{7.5}{8.7}$$

$$\therefore\quad \theta = \cos^{-1}\left(\frac{7.5}{8.7}\right)$$

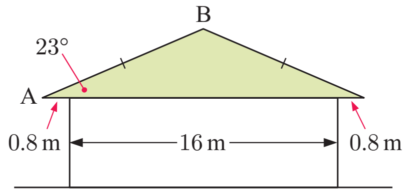
$$\therefore\quad \theta \approx 30.450^o$$

$\therefore$ the pitch is approximately $30\frac{1}{2}^o$.

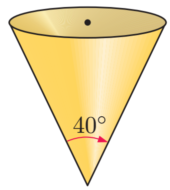
10 Find θ, the pitch of the roof.



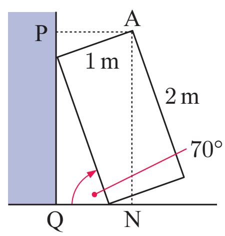
11 If the pitch of the given roof is 23^o, find the length of the timber beam [AB].



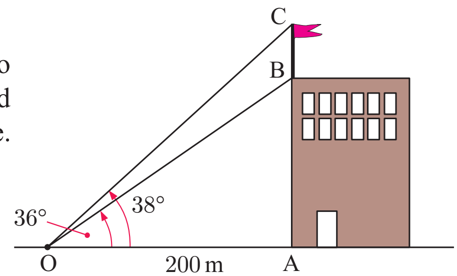
12 An open right-circular cone has a vertical angle measuring 40^o and a base radius of 30 cm.
Find the capacity of the cone in litres.



13 A refrigerator is tipped against a vertical wall so it can be serviced. It makes an angle of 70^o with the horizontal floor. How high is point A above the floor?



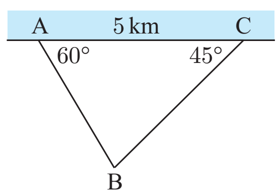
14 From an observer O, the angles of elevation to the bottom and the top of a flagpole are 36^o and 38^o respectively. Find the height of the flagpole.



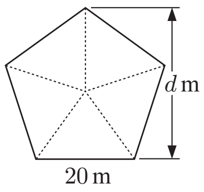
15 The angle of depression from the top of a 150 m high cliff to a boat at sea is 7^o. How much closer to the cliff must the boat move for the angle of depression to become 19^o?

16 A helicopter flies horizontally at 100 km h^{-1}. An observer notices that it takes 20 seconds for the helicopter to fly from directly overhead to being at an angle of elevation of 60^o. Find the height of the helicopter above the ground.

17 [AC] is a straight shore line and B is a boat out at sea. Find the shortest distance from the boat to the shore if A and C are 5 km apart.



18



A regular pentagonal garden plot is to be constructed with sides of length 20 m. Find the width of land d m required for the plot.

EXERCISE A

1 **a** $\sqrt{15}$ **b** 3 **c** 4 **d** 12 **e** 42
f $\sqrt{6}$ **g** $\sqrt{2}$ **h** $\sqrt{6}$

2 **a** $5\sqrt{2}$ **b** $-\sqrt{2}$ **c** $2\sqrt{5}$ **d** $8\sqrt{5}$ **e** $-2\sqrt{5}$
f $9\sqrt{3}$ **g** $-3\sqrt{6}$ **h** $3\sqrt{2}$

3 **a** $2\sqrt{2}$ **b** $2\sqrt{3}$ **c** $2\sqrt{5}$ **d** $4\sqrt{2}$ **e** $3\sqrt{3}$
f $3\sqrt{5}$ **g** $4\sqrt{3}$ **h** $3\sqrt{6}$ **i** $5\sqrt{2}$ **j** $4\sqrt{5}$
k $4\sqrt{6}$ **l** $6\sqrt{3}$

4 **a** $2\sqrt{3}$ **b** $8\sqrt{2}$ **c** $5\sqrt{6}$ **d** $10\sqrt{3}$ **e** $3\sqrt{3}$
f $-\sqrt{2}$

5 **a** $\dfrac{\sqrt{2}}{2}$ **b** $2\sqrt{3}$ **c** $\dfrac{7\sqrt{2}}{2}$ **d** $2\sqrt{5}$ **e** $5\sqrt{2}$
f $3\sqrt{6}$ **g** $4\sqrt{3}$ **h** $\dfrac{5\sqrt{7}}{7}$ **i** $2\sqrt{7}$ **j** $\sqrt{6}$

EXERCISE B

1 **a** 2.59×10^2 **b** 2.59×10^5 **c** 2.59×10^0
d 2.59×10^{-1} **e** 2.59×10^{-4} **f** 4.07×10^1
g 4.07×10^3 **h** 4.07×10^{-2} **i** 4.07×10^5
j 4.07×10^8 **k** 4.07×10^{-5}

2 **a** 1.495×10^{11} m **b** 3×10^{-4} mm **c** 1×10^{-3} mm
d 1.5×10^7 °C **e** 3×10^5

3 **a** 4000 **b** 500 **c** 2100 **d** 78 000
e 380 000 **f** 86 **g** 43 300 000 **h** 60 000 000

4 **a** 0.004 **b** 0.05 **c** 0.0021 **d** 0.000 78
e 0.000 038 **f** 0.86 **g** 0.000 000 433
h 0.000 000 6

5 **a** 0.000 000 9 m **b** 6 130 000 000
c 100 000 light years **d** 0.000 01 mm

6 **a** 1.64×10^{10} **b** 4.12×10^{-3} **c** 5.27×10^{-18}
d 1.36×10^2 **e** 2.63×10^{-6} **f** 1.73×10^9

7 **a** 1.30×10^5 km **b** 9.07×10^5 km **c** 9.47×10^7 km

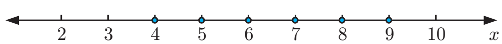
8 **a** 1.8×10^{10} m **b** 2.59×10^{13} m **c** 9.47×10^{15} m

EXERCISE C

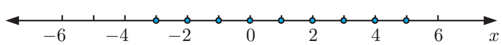
1 **a** The set of all real x such that x is greater than 5.
b The set of all integers x such that x is less than or equal to 3.
c The set of all y such that y lies between 0 and 6.
d The set of all integers x such that x is greater than or equal to 2, but less than or equal to 4. x is 2, 3 or 4.
e The set of all t such that t lies between 1 and 5.
f The set of all n such that n is less than 2 or greater than or equal to 6.

2 **a** $\{x \mid x > 2\}$ **b** $\{x \mid 1 < x \leqslant 5\}$
c $\{x \mid x \leqslant -2 \text{ or } x \geqslant 3\}$ **d** $\{x \mid -1 \leqslant x \leqslant 3,\ x \in \mathbb{Z}\}$
e $\{x \mid 0 \leqslant x \leqslant 5,\ x \in \mathbb{Z}\}$ **f** $\{x \mid x < 0\}$

3 **a**

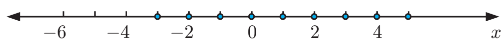


b

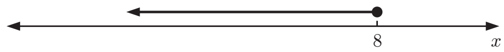


c

d



e



EXERCISE D

1 **a** $10x - 10$ **b** $9x$ **c** $5x + 5y$ **d** $8 - 8x$
e $12ab$ **f** cannot be simplified

2 **a** $22x + 35$ **b** $16 - 6x$ **c** $4a - 3b$
d $3x^3 - 16x^2 + 11x - 1$

3 **a** $18x^3$ **b** $\dfrac{a}{3b}$ **c** $4x^2$ **d** $24a^{10}$

EXERCISE E

1 **a** $x = 10$ **b** $x > 6$ **c** $x = \frac{4}{5}$ **d** $x = 51$
e $x < -10$ **f** $x = 14$ **g** $x \leqslant -9$ **h** $x = 18$
i $x = \frac{2}{3}$

2 **a** $x = 5,\ y = 2$ **b** $x = \frac{22}{3},\ y = \frac{8}{3}$ **c** $x = -2,\ y = 5$
d $x = \frac{45}{11},\ y = -\frac{18}{11}$ **e** no solution **f** $x = 66,\ y = -84$

EXERCISE F

1 **a** 16 **b** -6 **c** 16 **d** 18 **e** -2 **f** 2

2 **a** 2 **b** 3 **c** 6 **d** 6 **e** 5 **f** -1
g 1 **h** 5 **i** 4 **j** 4 **k** 2 **l** 2

3 **a** $x = \pm 3$ **b** no solution **c** $x = 0$
d $x = 4$ or -2 **e** $x = -1$ or 7 **f** no solution
g $x = 1$ or $\frac{1}{3}$ **h** $x = 0$ or 3 **i** $x = -2$ or $\frac{14}{5}$

EXERCISE G

1 **a** $2x^2 + 5x + 3$ **b** $3x^2 + 10x + 8$
c $10x^2 + x - 2$ **d** $3x^2 + x - 10$
e $-6x^2 + 17x + 14$ **f** $-6x^2 - 13x + 5$
g $15x^2 + 11x - 12$ **h** $15x^2 - 11x + 2$
i $2x^2 - 17x + 21$ **j** $4x^2 - 16x + 15$
k $-x^2 - 3x - 2$ **l** $-4x^2 - 2x + 6$

2 **a** $x^2 - 36$ **b** $x^2 - 64$ **c** $4x^2 - 1$ **d** $9x^2 - 4$
e $16x^2 - 25$ **f** $25x^2 - 9$ **g** $9 - x^2$ **h** $49 - x^2$
i $49 - 4x^2$ **j** $x^2 - 2$ **k** $x^2 - 5$ **l** $4x^2 - 3$

3 **a** $x^2 + 10x + 25$ **b** $x^2 + 14x + 49$ **c** $x^2 - 4x + 4$
d $x^2 - 12x + 36$ **e** $x^2 + 6x + 9$ **f** $x^2 + 10x + 25$
g $x^2 - 22x + 121$ **h** $x^2 - 20x + 100$ **i** $4x^2 + 28x + 49$
j $9x^2 + 12x + 4$ **k** $4x^2 - 20x + 25$ **l** $9x^2 - 42x + 49$

4 **a** $y = 2x^2 + 10x + 12$ **b** $y = 3x^2 - 6x + 7$
c $y = -x^2 + 6x + 7$ **d** $y = -x^2 - 4x - 15$
e $y = 4x^2 - 24x + 20$ **f** $y = -\frac{1}{2}x^2 - 4x - 14$
g $y = -5x^2 + 35x - 30$ **h** $y = \frac{1}{2}x^2 + 2x - 4$
i $y = -\frac{5}{2}x^2 + 20x - 40$

5 **a** $2x^2 + 12x + 19$ **b** $3x^2 + 3x - 16$ **c** $-x^2 + 6x - 6$
d $-x^2 - x + 25$ **e** $2x^2 - 16x + 33$ **f** $-3x + 4$ **g** $7x + 8$
h $7x^2 + 18x + 12$ **i** $-x^2 + 19x - 32$ **j** $7x^2 - 16x + 2$

EXERCISE H

1 **a** $3x(x+3)$ **b** $x(2x+7)$ **c** $2x(2x-5)$ **d** $3x(2x-5)$
e $(3x-5)(3x+5)$ **f** $(4x+1)(4x-1)$ **g** $2(x-2)(x+2)$
h $3(x+\sqrt{3})(x-\sqrt{3})$ **i** $4(x+\sqrt{5})(x-\sqrt{5})$ **j** $(x-4)^2$
k $(x-5)^2$ **l** $2(x-2)^2$ **m** $(4x+5)^2$ **n** $(3x+2)^2$
o $(x-11)^2$

2 **a** $(x+8)(x+1)$ **b** $(x+4)(x+3)$ **c** $(x-9)(x+2)$
d $(x+7)(x-3)$ **e** $(x-6)(x-3)$ **f** $(x+3)(x-2)$
g $-(x-2)(x+1)$ **h** $3(x-11)(x-3)$ **i** $-2(x+1)^2$
j $2(x+5)(x-2)$ **k** $2(x-8)(x+3)$ **l** $-2(x-6)(x-1)$
m $-3(x-1)^2$ **n** $-(x+1)^2$ **o** $-5(x-4)(x+2)$

3 **a** $(2x-3)(x+4)$ **b** $(3x+1)(x-2)$ **c** $(7x-2)(x-1)$
d $(3x-2)(2x+1)$ **e** $(2x-3)(2x+1)$ **f** $(5x-3)(2x+1)$
g $(2x+1)(x-6)$ **h** $(3x+7)(x-4)$ **i** $(4x+3)(2x-1)$
j $(5x+3)(2x-3)$ **k** $(3x-1)(x+8)$ **l** $(3x+2)(2x+1)$
m $-2(2x+3)(x-1)$ **n** $(6x+1)(2x-3)$
o $-3(2x+7)(x-2)$ **p** $-(3x-2)(3x-5)$
q $(4x-9)(2x+3)$ **r** $(4x+3)(3x+1)$ **s** $(6x+1)(2x+3)$
t $(5x-4)(3x-2)$ **u** $(7x+5)(2x-3)$

4 **a** $(x+4)(2x+1)$ **b** $(2-x)(5-3x)$
c $3(x+2)(2x+7)$ **d** $4(x+5)(2x+11)$
e $2x(x+3)$ **f** $5(x+3)$
g $(x-2)(3x+26)$ **h** $(x-1)(2x-1)$

5 **a** $(x+7)(x-1)$ **b** $(x+1)(3-x)$ **c** $12(x+1)$
d $-4x(x+4)$ **e** $(3x+2)(x+4)$ **f** $h(2x+h)$
g $-12(x+1)$ **h** $-5(3x-4)(x-4)$
i $-3(x+9)(5x+9)$

EXERCISE I

1 **a** $x=b-a$ **b** $x=\dfrac{b}{a}$ **c** $x=\dfrac{d-a}{2}$
d $x=t-c$ **e** $x=\dfrac{20-2y}{5}$ **f** $x=\dfrac{12-3y}{2}$
g $x=\dfrac{d-3y}{7}$ **h** $x=\dfrac{c-by}{a}$ **i** $x=\dfrac{y-c}{m}$

2 **a** $z=\dfrac{b}{ac}$ **b** $z=\dfrac{a}{d}$ **c** $z=\dfrac{2d}{3}$ **d** $z=\pm\sqrt{2a}$

3 **a** $a=\dfrac{F}{m}$ **b** $r=\dfrac{C}{2\pi}$ **c** $d=\dfrac{V}{lh}$ **d** $K=\dfrac{b}{A}$

4 **a** $r=\sqrt{\dfrac{A}{\pi}}$ **b** $x=\sqrt[5]{aN}$ **c** $r=\sqrt[3]{\dfrac{3V}{4\pi}}$
d $x=\sqrt[3]{\dfrac{n}{D}}$

5 **a** $a=d^2n^2$ **b** $l=25T^2$ **c** $a=\pm\sqrt{b^2+c^2}$
d $l=\dfrac{gT^2}{4\pi^2}$ **e** $a=\dfrac{P}{2}-b$ **f** $h=\dfrac{A-\pi r^2}{2\pi r}$
g $r=\dfrac{E}{I}-R$ **h** $q=p-\dfrac{B}{A}$

6 **a** $a=\dfrac{d^2}{2kb}$ **b** 1.29 **7** **a** $r=\sqrt[3]{\dfrac{3V}{4\pi}}$ **b** 2.122 cm

8 **a** $t=\sqrt{\dfrac{2S}{a}}$ **b** 15.81 s

9 **a** $v=\dfrac{uf}{u-f}$ **b** **i** 9.52 cm **ii** 10.9 cm

10 **a** $v=\sqrt{c^2\left(1-\dfrac{{m_0}^2}{m^2}\right)}=\dfrac{c}{m}\sqrt{m^2-{m_0}^2}$
b $v=\frac{\sqrt{8}}{3}c$ **c** $2.998\times10^8\ \text{m s}^{-1}$

EXERCISE J

1 **a** $\dfrac{15+x}{5}$ **b** $\dfrac{x+3}{x}$ **c** $\dfrac{x+4}{2}$
d $\dfrac{14-x}{4}$ **e** $\dfrac{8x-2}{15}$ **f** $\dfrac{4x+17}{12}$

2 **a** $\dfrac{x+5}{x+2}$ **b** $\dfrac{11-2x}{x-4}$ **c** $\dfrac{1-3x}{x-1}$
d $\dfrac{5x+2}{x+1}$ **e** $\dfrac{2x+3}{x+1}$ **f** $\dfrac{x+3}{1-x}$

3 **a** $\dfrac{7x^2-6x-25}{(2x-5)(x-2)}$ **b** $\dfrac{-1}{(x-2)(x-3)}$
c $\dfrac{8x^2+6x+8}{x^2-16}$ **d** $\dfrac{3x^2+3x+13}{(x-3)(2x+1)}$

EXERCISE K.1

1 **Hint:** Consider Δs AEC, BDC

2 **Hint:** Let M be on [AB] so that [PM] $\perp$ [AB].
Let N be on [AC] so that [PN] $\perp$ [AC].
Join [PM], [PN] and consider the two triangles formed.

EXERCISE K.2

1 **a** $x=2.4$ **b** $x=2.8$ **c** $x=3\frac{3}{11}$ **d** $x=6\frac{2}{3}$
e $x=7$ **f** $x=7.2$

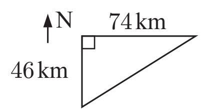
2 1.35 m tall

EXERCISE L.1

1 **a** $x=0.663$ **b** $x=4.34$ **c** $x=2.23$

2 **a** 4.54 m **b** 4.17 m **3** 237 m

4 **a**



b 87.1 km

EXERCISE L.2

1 50.3 m **2** 17.3 cm **3** 8.60 m **4** 53.4 m **5** 44.4 km

EXERCISE M

1 **a** $\sqrt{13}$ units **b** $\sqrt{34}$ units **c** $\sqrt{20}$ units **d** $\sqrt{10}$ units

2 **a** (2, 3) **b** (2, -1) **c** $(3\frac{1}{2}, 1\frac{1}{2})$ **d** $(2, -2\frac{1}{2})$

3 **a** $\frac{2}{3}$ **b** $-\frac{2}{5}$ **c** 1 **d** 0 **e** -4 **f** undefined

4 **a** 2 **b** 3 **c** undefined **d** 0 **e** -4 **f** $\frac{1}{4}$

5 **a** parallel, gradients $\frac{2}{5}$ **b** parallel, gradients $-\frac{1}{7}$
c perpendicular, gradients $\frac{1}{2}, -2$
d neither, gradients $-4, \frac{1}{3}$ **e** neither, gradients $\frac{2}{7}, \frac{1}{5}$
f perpendicular, gradients $2, -\frac{1}{2}$

6 **a** $-\frac{4}{3}$ **b** $-\frac{3}{11}$ **c** $-\frac{1}{4}$ **d** 3 **e** $\frac{1}{5}$ **f** undefined

7 **a** $y=2x-7$ **b** $y=-2x+4$ **c** $y=3x-15$
d $y=-3x+4$ **e** $y=-4x+9$ **f** $y=x+5$

8 **a** $3x-2y=4$ **b** $3x+2y=11$ **c** $x-3y=4$
d $4x+y=6$ **e** $3x-y=0$ **f** $4x+9y=-2$

9 **a** $x-3y=-3$ **b** $5x-y=1$ **c** $x-y=3$
d $4x-5y=10$ **e** $x-2y=-1$ **f** $2x+3y=-5$

10 **a** $y=-2$ **b** $x=6$ **c** $x=-3$

11

	Equation of line	*Gradient*	*x-int.*	*y-int.*
a	$2x-3y=6$	$\frac{2}{3}$	3	-2
b	$4x+5y=20$	$-\frac{4}{5}$	5	4
c	$y=-2x+5$	-2	$\frac{5}{2}$	5
d	$x=8$	undef.	8	no y-int.
e	$y=5$	0	no x-int.	5
f	$x+y=11$	-1	11	11
g	$4x+y=8$	-4	2	8
h	$x-3y=12$	$\frac{1}{3}$	12	-4

12 **a** yes **b** no **c** yes

13 **a** $(4, 2)$ **b** $(-2, 3)$ **c** $(-3, 6)$ **d** $(4, 0)$
e parallel lines do not meet **f** coincident lines

14 **a** $y=-4$ **b** $x=5$ **c** $x=-1$ **d** $y=2$
e $y=0$ **f** $x=0$

15 **a** $3x-4y=-19$ **b** $x-y=7$ **c** $y=3x$
d $2x+3y=19$ **e** $x-2y=5$ **f** $x+3y=13$

16 **a** $x-8y=-83$ **b** $8x+y=41$
c $9x-2y=23$ for $5 \leqslant x \leqslant 7$ **d** $(8, 11\frac{3}{8})$

17 **a** $x-3y=-16$ **b** $2x-y=-3$ **c** $x=5$

18 $(4\frac{3}{4}, 8)$ **19** **a** $(2\frac{1}{3}, 7)$ **b** 8.03 km **c** yes (6.16 km)

20 $(13.41, 8)$ or $(-5.41, 8)$

21 **a** **Hint:** Use the distance formula to find the distance from the centre of the circle to point P.
b **i** $(x-4)^2+(y-3)^2=25$ **ii** $(x+1)^2+(y-5)^2=4$
iii $x^2+y^2=100$ **iv** $(x-1)^2+(y-3)^2=8$

22 **a** centre $(1, 3)$, radius 2 units
b centre $(0, -2)$, radius 4 units
c centre $(0, 0)$, radius $\sqrt{7}$ units

23 **a** centre $(2, -3)$, radius $\sqrt{20}$ units
b **Hint:** Substitute $(4, 1)$ into equation of circle.
c $x+2y=6$

24 $(3, 3)$

EXERCISE N.1

1 **a** 3, $\sin\theta=\frac{3}{5}$, $\cos\theta=\frac{4}{5}$, $\tan\theta=\frac{3}{4}$
b 12, $\sin\theta=\frac{5}{13}$, $\cos\theta=\frac{12}{13}$, $\tan\theta=\frac{5}{12}$
c $\sqrt{11}$, $\sin\theta=\frac{5}{6}$, $\cos\theta=\frac{\sqrt{11}}{6}$, $\tan\theta=\frac{5}{\sqrt{11}}$
d $\sqrt{5}$, $\sin\theta=\frac{2}{\sqrt{5}}$, $\cos\theta=\frac{1}{\sqrt{5}}$, $\tan\theta=2$
e $\sqrt{52}$, $\sin\theta=\frac{4}{\sqrt{52}}$, $\cos\theta=\frac{6}{\sqrt{52}}$, $\tan\theta=\frac{2}{3}$
f $\sqrt{15}$, $\sin\theta=\frac{7}{8}$, $\cos\theta=\frac{\sqrt{15}}{8}$, $\tan\theta=\frac{7}{\sqrt{15}}$

2 **a** $\sin\theta=\frac{\sqrt{3}}{2}$, $\tan\theta=\sqrt{3}$ **b** $\cos\alpha=\frac{\sqrt{5}}{3}$, $\tan\alpha=\frac{2}{\sqrt{5}}$
c $\sin\beta=\frac{4}{5}$, $\cos\beta=\frac{3}{5}$

3 **a** $\sin\theta=\dfrac{b}{c}$, $\cos\theta=\dfrac{a}{c}$, $\tan\theta=\dfrac{b}{a}$

4 **a** **i** $\dfrac{a}{b}$ **ii** $\dfrac{c}{b}$ **iii** $\dfrac{c}{b}$ **iv** $\dfrac{a}{b}$
b **i** complement **ii** complement

5 **a** $\sqrt{2}$ **b** $\sin 45^o=\frac{1}{\sqrt{2}}$, $\cos 45^o=\frac{1}{\sqrt{2}}$, $\tan 45^o=1$

6 **a** $\widehat{ABN}=60^o$, $\widehat{BAN}=30^o$ **b** BN $=1$, AN $=\sqrt{3}$
c **i** $\sin 60^o=\frac{\sqrt{3}}{2}$, $\cos 60^o=\frac{1}{2}$, $\tan 60^o=\sqrt{3}$
ii $\sin 30^o=\frac{1}{2}$, $\cos 30^o=\frac{\sqrt{3}}{2}$, $\tan 30^o=\frac{1}{\sqrt{3}}$

EXERCISE N.2

1 **a** $x=17.2$ **b** $x=257$ **c** $x=15.1$
d $x=7.10$ **e** $x=554$ **f** $x=457$

2 **a** $\theta=69.5^o$ **b** $\theta=76.2^o$ **c** $\theta=60.0^o$
d $\theta=73.4^o$ **e** $\theta=19.5^o$ **f** $\theta=77.9^o$
g $\theta=9.06^o$ **h** $\theta=34.7^o$

3 **a** $\theta=56.4$ **b** $\alpha=4.76$ **c** $\beta=48.2$

4 **a** AC $=6.40$ m, $\widehat{A}=38.7^o$, $\widehat{C}=51.3^o$
b $\widehat{R}=39^o$, PQ $=8.10$ m, PR $=12.9$ m

5 **a** $x=2.65$, $\theta=37.1$
b $x=6.16$, $\theta=50.3$, $y=13.0$

EXERCISE N.3

1 **a** $x=4.13$ **b** $\alpha=75.5$ **c** $\beta=41.0$
2 **a** $\theta=36.9$ **b** $r=11.3$ **c** $\alpha=61.9$ **3** 7.99 cm
4 89.2^o **5** 47.2^o

EXERCISE N.4

1 22.4^o **2** 11.8 cm **3** 119^o **4** 36.5 cm
5 **a** $x=3.44$ **b** $\alpha=51.5$ **6** 129^o

EXERCISE N.5

1 18.3 m **2** **a** 371 m **b** 1.62 km **3** 159 m **4** 1.58^o
5 angle of elevn. $=26.4^o$, angle of depn. $=26.4^o$
6 418.5 m **7** 111 m **8** 72.0 m
9 **a** 16.2 m s^{-1} **b** 11.5^o **10** $\theta=12.6$ **11** 9.56 m
12 77.7 litres **13** 2.22 m **14** 10.95 m **15** 786 m **16** 962 m
17 3.17 km **18** 30.8 m