Theorem

$$\vec{a}\bullet\vec{b} = \left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$$

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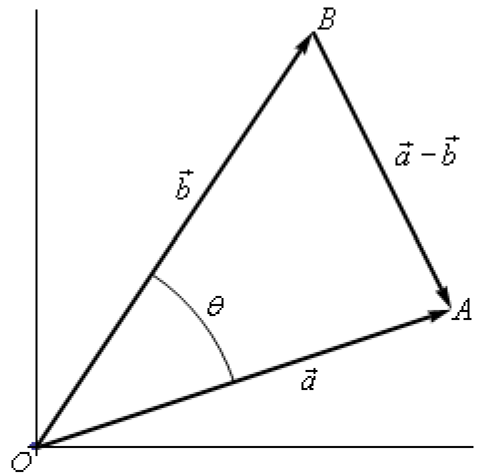
(2)

Proof

Let’s give a modified version of the sketch above.



The three vectors above form the triangle *AOB* and note that the length of each side is nothing more than the magnitude of the vector forming that side.

The Law of Cosines tells us that,

$$\left\|\vec{a}-\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$$

$\left\|\vec{a}-\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\| \left\|\vec{b}\right\| \cos\theta$

Also using the properties of dot products we can write the left side as,

$$\begin{aligned}\left\|\vec{a}-\vec{b}\right\|^2 &= \left(\vec{a}-\vec{b}\right)\bullet\left(\vec{a}-\vec{b}\right)\\ &= \vec{a}\bullet\vec{a} - \vec{a}\bullet\vec{b} - \vec{b}\bullet\vec{a} + \vec{b}\bullet\vec{b}\\ &= \left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2\end{aligned}$$

$$\left\|\vec{a}-\vec{b}\right\|^2 = \left(\vec{a}-\vec{b}\right)\bullet\left(\vec{a}-\vec{b}\right)$$
$$= \vec{a}\bullet\vec{a} - \vec{a}\bullet\vec{b} - \vec{b}\bullet\vec{a} + \vec{b}\bullet\vec{b}$$
$$= \left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2$$

Our original equation is then,

$$\left\|\vec{a}-\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$\left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$-2\vec{a}\bullet\vec{b} = -2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$\vec{a}\bullet\vec{b} = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$

$$\left\|\vec{a}-\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$\left\|\vec{a}\right\|^2 - 2\vec{a}\bullet\vec{b} + \left\|\vec{b}\right\|^2 = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 - 2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$-2\vec{a}\bullet\vec{b} = -2\left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$
$$\vec{a}\bullet\vec{b} = \left\|\vec{a}\right\|\left\|\vec{b}\right\|\cos\theta$$

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The formula from this theorem is often used not to compute a dot product but instead to find the angle between two vectors. Note as well that while the sketch of the two vectors in the proof is for two dimensional vectors the theorem is valid for vectors of any dimension (as long as they have the same dimension of course).

Let's see an example of this.

Example 2 Determine the angle between $\vec{a} = \langle 3,-4,-1\rangle$ $\vec{a} = \langle 3,-4,-1\rangle$ and $\vec{b} = \langle 0,5,2\rangle$ $\vec{b} = \langle 0,5,2\rangle$.

Solution

We will need the dot product as well as the magnitudes of each vector.

$$\vec{a}\bullet\vec{b} = -22 \qquad \left\|\vec{a}\right\| = \sqrt{26} \qquad \left\|\vec{b}\right\| = \sqrt{29}$$

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The angle is then,

$$\cos\theta = \frac{\vec{a}\bullet\vec{b}}{\|\vec{a}\|\,\|\vec{b}\|} = \frac{-22}{\sqrt{26}\sqrt{29}} = -0.8011927$$

$$\theta = \cos^{-1}\left(-0.8011927\right) = 2.5 \text{ radians} = 143.24 \text{ degrees}$$

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The dot product gives us a very nice method for determining if two vectors are perpendicular and it will give another method for determining when two vectors are parallel. Note as well that often we will use the term **orthogonal** in place of perpendicular.

Now, if two vectors are orthogonal then we know that the angle between them is 90 degrees. From **(2)** this tells us that if two vectors are orthogonal then,

$$\vec{a}\bullet\vec{b} = 0$$

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Likewise, if two vectors are parallel then the angle between them is either 0 degrees (pointing in the same direction) or 180 degrees (pointing in the opposite direction). Once again using **(2)** this would mean that one of the following would have to be true.

$$\vec{a}\bullet\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\ \left(\theta = 0^{\circ}\right) \qquad \text{OR}$$

$$\vec{a}\bullet\vec{b} = \|\vec{a}\|\,\|\vec{b}\|\ \left(\theta = 0^{\circ}\right) \qquad \text{OR}$$
$$\vec{a}\bullet\vec{b} = -\|\vec{a}\|\,\|\vec{b}\|\ (\theta$$