6 SEQUENCES AND SERIES

Objectives

After studying this chapter you should

- be able to work with both finite and infinite series;
- understand, and be able to apply, the method of proof by mathematical induction;
- be able to use the method of differences to sum finite series, and extend its use to infinite series;
- know how to obtain Maclaurin series for well known functions, including the general binomial expansion;
- appreciate that some of the standard series are valid for all real values of *x*, while others are only valid for specified ranges of values of *x*;
- be able to determine general terms in these standard cases.

6.0 Introduction and revision

In *Pure Mathematics*, Chapter 13, you were introduced to the notion of a sequence, and its related series. You will also have encountered the use of the $\sum$ notation as a shorthand for writing out series with a large number of terms (possibly infinitely many). The first section of this chapter will remind you of the essential points that you will need in order to develop further the work in this area.

Definition

A **sequence** is simply an ordered list $u_1, u_2, u_3, \ldots, u_n$, of numbers (or terms). This is often abbreviated to $\{u_n\}$. For our purposes each term u_n is usually given in one of two ways:

(i) as a function of the preceding term(s), or

(ii) as a function of its position in the sequence.

Example

The sequence $\{u_n\}$ defined by

$$u_1 = 1 \text{ and } u_n = (u_1 u_2 \ldots u_{n-1}) + 1 \text{ for } n \geq 2$$

is 1, 2, 3, 7, 43, 1807, ...,

where each term (after the first) is one greater than the product of all the previous terms of the sequence.

You will appreciate however that, in such cases, should you wish to know (say) u_{100}, the hundredth term of this sequence, then you would first need to know each of the preceding 99 terms. It is preferable, then, to have sequences given in the second way, with each term defined as a function of *n*, its position in the sequence.

Example

One of the most famous sequences of all is the Fibonacci sequence $\{F_n\}$ which is defined by

$$F_1 = 1, \quad F_2 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

This sequence begins

1, 1, 2, 3, 5, 8, 13, 21, ...

The general solution of sequences defined in this way is not within the scope of this course, but (for the interested reader) each term of the Fibonacci sequence is actually given by

$$F_n = \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right\}$$

Activity 1

Check that the formula above for the Fibonacci sequence does, in fact, give the first three terms.

The activity above shows that an analytic approach is not always the most useful. It is much easier to find, say, F_{20} from adding the terms rather than by using the formula.

6.1 Arithmetic and geometric sequences and series

The sequence defined by

$$u_1 = a \text{ and } u_n = u_{n-1} + d \text{ for } n \geq 2$$

begins

$$a,\ a+d,\ a+2d, \ldots$$

and you should recognise this as the **arithmetic sequence** with first term a and common difference d.

The nth term (i.e. the solution) is given by $u_n = a + (n-1)d$.

The arithmetic series with n terms,

$$a + (a+d) + (a+2d) + \ldots + [a + (n-1)d],$$

has sum

$$\boxed{S_n = \frac{n}{2}[2a + (n-1)d]}$$

or

$$S_n = \frac{n}{2}(\text{first } + \text{ last})$$

and these results should be well known to you.

Example

An arithmetic series has the property that the sum of the first ten terms is half the sum of the **next** ten terms. Also its 100th term is 95. Find the first term and common difference.

Solution

Let the first term be a and the common difference be d. Then the sums of the first ten terms and the next ten terms are

$$\frac{10}{2}(a + (a+9d)) \text{ and } \frac{10}{2}((a+10d) + (a+19d))$$

Since the former is half the latter we deduce that

$$2a + 9d = \frac{1}{2}(2a + 29d)$$

or $\qquad 2a = 11d \qquad (1)$

Also, as the 100th term is 95, we know that

$$a + 99d = 95 \qquad (2)$$

From (1) $99d = 18a$ and so in (2)

$$a + 18a = 19a = 95$$

Hence $\quad a = 5$ and $d = \dfrac{10}{11}$.

Another important sequence defined by

$$u_1 = a \text{ and } u_n = ru_{n-1} \text{ for } n \geq 2,$$

begins

$$a,\ ar,\ ar^2, \ldots$$

This is the **geometric sequence** with first term a and common ratio r.

The nth term is given by $u_n = ar^{n-1}$.

The geometric series with n terms,

$$a + ar + ar^2 + \ldots + ar^{n-1}$$

has sum

$$\boxed{S_n = \frac{a(1-r^n)}{1-r} \text{ or } \frac{a(r^n-1)}{r-1}} \qquad \text{for } r \neq 1$$

Note that a **series** is the sum of a number of terms of a **sequence**. The terms 'arithmetic progression' (A.P.) and 'geometric progression' (G.P.) are not preferred here as the word 'progression' is used loosely in respect of both a sequence and a series. However, you should recognise the use of these terms and their abbreviations, since they are in common usage.

6.2 The sigma notation

When writing series, the shorthand $\sum$ notation is used to represent the sum of a number of terms having a common form.

The series $f(1) + f(2) + \ldots + f(n-1) + f(n)$ would be written

$$\sum_{r=1}^{n} f(r).$$

What other way could the series be written using $\sum$ notation?

The function f is the common form that each term of the series takes; r is called the **summation index**, being the variable quantity from term to term. The '$r=1$' below the sigma indicates the first value taken by r, and the 'n' above the sigma denotes the final term taken by r.

Note that, while it is customary to take $r=1$ for the first term, it is by no means essential. The above series would equally well be written as

$$\sum_{r=0}^{n-1} f(r+1)$$

In fact, later on in the chapter you will encounter series which, for convenience, will start with $r=0$.

Also, the choice of the letter 'r' is unimportant:

$$\sum_{r=1}^{n} f(r), \ \sum_{k=1}^{n} f(k) \text{ and } \sum_{i=1}^{n} f(i)$$

all represent the series $f(1)+f(2)+\ldots+f(n)$.

You should know the following results relating to the $\sum$ notation:

(1) $$\sum_{r=1}^{n}\{f(r)+g(r)\} = \{f(1)+g(1)\}+\{f(2)+g(2)\}+\ldots+\{f(n)+g(n)\}$$

$$=\{f(1)+f(2)+\ldots+f(n)\}+\{g(1)+g(2)+\ldots+g(n)\}$$

$$=\sum_{r=1}^{n} f(r)+\sum_{r=1}^{n} g(r)$$

(2) $$\sum_{r=1}^{n} af(r) = af(1)+af(2)+\ldots+af(n)$$ where a is some constant

$$=a\{f(1)+f(2)+\ldots+f(n)\}$$

$$=a\sum_{r=1}^{n} f(r)$$

Note that $\sum_{r=1}^{n} n\,f(r) = n\sum_{r=1}^{n} f(r)$ also, since n is a fixed quantity, not a variable, in the summation.

(3) (a) $$\sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

(b) $$\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$$

(c) $$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2 \quad \text{or} \quad \left[\frac{n(n+1)}{2}\right]^2$$

(d) $$\sum_{r=1}^{n} 1 = \underbrace{(1+1+\ldots+1)}_{\leftarrow\ n \text{ times}\ \rightarrow} = n$$

Since a '1' is wanted for each of $r = 1, 2, \ldots, n$

This result is easily overlooked, and often incorrectly written as

$$\sum_{r=1}^{n} 1 = 1$$

Example

Show that $\sum_{r=1}^{n}\left(6r^2 + 4r - 1\right) = n(n+2)(2n+1)$.

Solution

$$\sum_{r=1}^{n}\left(6r^2 + 4r - 1\right) = \sum_{r=1}^{n} 6r^2 + \sum_{r=1}^{n} 4r - \sum_{r=1}^{n} 1 \qquad \text{by result (1)}$$

$$= 6\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 1 \qquad \text{by result (2)}$$

$$= 6.\frac{n}{6}(n+1)(2n+1) + 4.\frac{n}{2}(n+1) - n \qquad \text{by result (3)}$$

$$= n(n+1)(2n+1) + 2n(n+1) - n$$

$$= n\left(2n^2 + 3n + 1 + 2n + 2 - 1\right)$$

$$= n\left(2n^2 + 5n + 2\right)$$

$$= n(n+2)(2n+1)$$

Exercise 6A

1. A geometric series has first term 4 and second term 7. Giving your answer to three significant figures, find the sum of the first twenty terms of the series. (AEB)
2. The first term of an arithmetic series is –13 and the last term is 99. The sum of the series is 1419. Find the number of terms and the common difference. Find also the sum of all the positive terms of the series. (AEB)
3. An arithmetic series has first term 7 and second term 11.
   (a) Find the 17th term.
   (b) Show that the sum of the first 35 terms is equal to the sum of the next 15 terms. (AEB)
4. The first three terms of a geometric series are $2x, x-8$ and $2x+5$ respectively. Find the possible values of x. (AEB)
5. An arithmetic series has first term $\ln 2$ and common difference $\ln 4$. Show that the sum S_n of the first n terms is $n^2 \ln 2$.

   Find the least value of n for which S_n is greater than fifty times the nth term. (AEB)
6. A geometric series has first term 4 and common ratio r, where $0<r<1$. Given that the first, second and fourth terms of this geometric series form three successive terms of an arithmetic series, show that $r^3-2r+1=0$. Find the value of r. (AEB)
7. Given that $S(n)=n^2+4n$, write down an expression for $S(n-1)$ and simplify $S(n)-S(n-1)$.

   By considering the result above, show that the sequence is arithmetic and state the first term and the common difference.
   Determine the least number of terms required for the sum of this arithmetic series to exceed $10\,000$. (AEB)
8. Show that $\sum_{r=1}^{n} r(r+1)=\frac{n}{3}(n+1)(n+2)$.
9. Use the formulae for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$ to

   show that $\sum_{r=1}^{n}\left(4r^3-3r^2+r\right)=An^3(n+1)$, for some constant A to be found.
10. Express the kth term of the series

    $$S=1.3.4+2.5.7+3.7.10+4.9.13+\dots$$

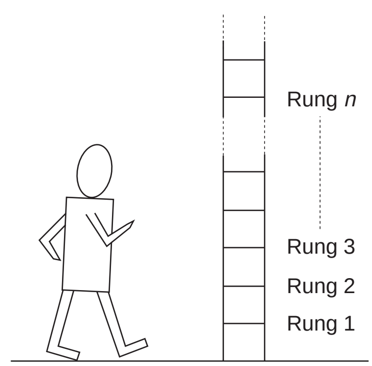
    as a function of k. Hence show that the sum of the first n terms of S is

    $$\frac{n}{6}(n+1)\left(9n^2+19n+8\right)$$
11. Find the formula for the sum of n terms of the series $4^2+7^2+10^2+13^2+\dots$
12. The rth term of a finite series is u_r, and the sum of n terms is denoted by S_n, so that $S_n=\sum_{r=1}^{n} u_r$.

    If $S_n=3n^2+5n$, express u_r as a function of r and find $\sum_{r=n}^{2n} u_r$.

6.3 Proof by induction

An inventor builds a climbing 'robot' which is designed to be able to climb any ladder which has equally spaced rungs, no matter how long it may be. (Being solar-powered, it can continue indefinitely if necessary.) The inventor tests it on a variety of ladders: on some of them, the robot succeeds; on others it does not.



Given any ladder, what conditions need to be satisfied for the robot to be able to climb the ladder?

You will see that there are, in fact, just two:

(a) In the first instance, the robot needs to be able to get on the ladder (presumably at the first rung); and

(b) if the robot is on any given rung (say rung k – as rungs are equally spaced, it does not matter where rung k is on the ladder), then it has to be able to get on to the next rung after this (rung $k+1$).

If the robot's programming and construction enable it to satisfy both these conditions, then it can climb as far up the ladder (ad infinitum if necessary) as its inventor could wish for; since by condition (a) the robot can get on the ladder (at rung 1). Then, by condition (b) it can get on to rung 2. By (b) again, it can get to rung 3, hence to rung 4, hence to rung 5, etc., etc. Thus the robot can reach rung n for **any** positive integer n.

No, the authors have not gone senile! The above is actually an illustration of a very powerful technique called **proof by induction**. The method is a cunning means of proving the truth of some statement, or formula, that is found by experimental means (for instance) but which, without a general proof, is only **known** to be true for certain values of the variable concerned.

For instance, the result

$$\sum_{r=1}^{n} r = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

arises from the sum to n terms of an arithmetic series, and has already been proved.

What about the result $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$**?**

Where did that come from?

One way, is to first note that the formula for $\sum_{r=1}^{n} r$ is a quadraticin n, and so assume that the formula for $\sum_{r=1}^{n} r^2$ is a cubic in n, say

$$an^3 + bn^2 + cn + d$$

Then use the values found in the cases when $n = 1, 2, 3$ and 4 to set up and solve a system of four simultaneous equations in the four unknowns a, b, c, d (which turn out to be $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{6}$ and 0 respectively).

However, this only **proves** that such an expression fits the bill in these four cases: namely $n = 1, 2, 3$ and 4. You could try it out for $n = 5, 6, 7, \ldots,$ as far as you liked. You could convince yourself that this expression for the sum of the squares of the first n positive integers just has to be true for every positive integer n; but you would have proved it in only a few (or even many) particular cases.

Try the robot approach:

when $n = 1$, $\sum_{r=1}^{n} r^2 = 1^2 = 1$,

while the formula with $n = 1$ gives

$$\tfrac{1}{6}(1)(1+1)(2\times1+1) = \tfrac{1}{6}\times1\times2\times3 = 1$$

also.

Hence the formula is true when $n = 1$. (The robot is on the ladder at rung 1.)

Next, **assume** that

$$\sum_{r=1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$$

(That is, the robot is on the ladder (somewhere) at rung *k*. Now what about rung $k+1$?)

Then it follows that

$$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$$

$$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2 \qquad \text{(the first term was assumed above)}$$

$$= \frac{k}{6}(k+1)(2k+1) + \frac{6}{6}(k+1)^2 \qquad \text{(common denominator)}$$

$$= \frac{(k+1)}{6}\left\{k(2k+1) + 6(k+1)\right\} \qquad \text{(factorising)}$$

$$= \frac{(k+1)}{6}\left(2k^2 + 7k + 6\right)$$

$$= \frac{(k+1)}{6}(k+2)(2k+3)$$

Now notice that this expression is

$$\frac{1}{6}(k+1)([k+1]+1)(2[k+1]+1),$$

which is precisely the formula expected, but with $n = k+1$.

So **if** the formula assumed when $n = k$ is true **then**, by adding on the $(k+1)$th term, it must also be true when $n = k+1$, whatever *k* is. (The robot can get from rung *k* to rung $k+1$.)

In itself, this step of the process is a big IF. But the formula **is** true when $n=1$, so this 'stepping up' bit proves it must be true when $n=2$ also. Since it is true for $n=2$ (which it now is known to be) then it must be true for $n=3$ as well; and then for $n=4$, $n=5, \ldots$ and for all positive integers n.

Remember that the 'stepping up' part of the proof relies on an assumption (called the **induction hypothesis**) and it is important, therefore, to use the word 'assume' (or 'suppose') otherwise there is a large amount of written explanation to do at the end of this proof in order to finalise matters conclusively. The proof is clinched by showing that the whole process starts in the first place.

Example

Use mathematical induction to prove that, for all positive integers n,

$$2.2+3.2^2+\ldots+(n+1).2^n=n.2^{n+1}$$

Solution

When $n=1$, LHS $=2.2=4$

and RHS $=1.2^{1+1}=1.4=4$

and the statement is true when $n=1$. (starting step)

Assume that the formula is true for $n=k$; that is

$$2.2+3.2^2+\ldots+(k+1).2^k=k.2^{k+1} \qquad \text{(induction hypothesis)}$$

Then, when $n=k+1$,

$$2.2+3.2^2+\ldots+(k+1).2^k+(k+2).2^{k+1}$$

$$=\underbrace{\left\{2.2+3.2^2+\ldots+(k+1).2^k\right\}}+(k+2).2^{k+1} \qquad ((k+1)\text{ term added})$$

$$= \qquad k.2^{k+1} \qquad +(k+2).2^{k+1} \qquad \text{(by the hypothesis)}$$

$$=2^{k+1}(k+k+2)$$

$$=2^{k+1}(2k+2)$$

$$=2^{k+1}.2(k+1)$$

$$=(k+1).2^{k+2}$$

which is the required formula with k replaced by $k+1$. Hence if the statement is true for $n=k$, then it is also true for $n=k+1$.

By induction $2.2+3.2^2+\ldots+(n+1).2^n = n.2^{n+1}$ is true for all positive integers n.

Note: because it is easy to decide what the final expression ('the formula for $n=k$ with k replaced by $k+1$') should be in advance, many students 'fiddle' it into existence, or simply write it straight down, without showing the necessary working to demonstrate that it does indeed arise as a result of adding on the $(k+1)$th term to the assumed sum-to-*k*-terms. Be careful to show your working!

The following example illustrates the minimal amount of working that needs to be put down in order to clinch an inductive proof.

Example

Prove by induction that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

for all positive integers n.

Solution

For $n=1$, LHS $= \sum_{r=1}^{1} \frac{1}{r(r+1)(r+2)} = \frac{1}{1\times 2\times 3} = \frac{1}{6}$,

while RHS$= \frac{1}{4} - \frac{1}{2\times 2\times 3} = \frac{1}{4} - \frac{1}{12} = \frac{1}{6}$ also.

Now assume that

$$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(k+1)(k+2)}$$

Then

$$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{(k+1)(k+2)}\left\{\frac{1}{2} - \frac{1}{k+3}\right\}$$

$$= \frac{1}{4} - \frac{1}{(k+1)(k+2)}\left\{\frac{k+3-2}{2(k+3)}\right\}$$

$$= \frac{1}{4} - \frac{(k+1)}{2(k+1)(k+2)(k+3)}$$

$$= \frac{1}{4} - \frac{1}{2([k+1]+1)([k+1]+2)}, \text{ as required.}$$

Proof follows by induction.

Exercise 6B

Use mathematical induction to prove the following results for all positive integers n:

1. $\sum_{r=1}^{n} r^3 = \frac{1}{4}n^2(n+1)^2$

2. $\sum_{r=1}^{n} r(r+3) = \frac{1}{3}n(n+1)(n+5)$

3. $\sum_{r=1}^{n} (2r-1)(2r+1) = \frac{1}{2} + \frac{1}{6}(2n-1)(2n+1)(2n+3)$

4. $\sum_{r=1}^{n} \frac{2}{(r+1)(r+2)(r+3)} = \frac{1}{6} - \frac{1}{(n+2)(n+3)}$

5. $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$

6. $\sum_{r=1}^{n} r \times 2^{r-1} = (n-1)\times 2^n + 1$

*7. $\cos x + \cos 3x + \cos 5x + \ldots + \cos(2n-1)x = \frac{\sin 2nx}{2\sin x}$

8. $\sum_{r=1}^{n} r(r+1)\left(\frac{1}{2}\right)^{r-1} = 16 - (n^2+5n+8)\left(\frac{1}{2}\right)^{n-1}$

9. $\sum_{r=1}^{n} (-1)^{r+1} r^2 = (-1)^{n+1} \times \frac{n(n+1)}{2}$

6.4 The difference method

The process of proof by induction, whilst being a powerful mathematical tool, has the disadvantage that, in order to employ it, you really need to have the answer (or something you strongly suspect to be the answer) to begin with.

There are, however, direct methods of proof available in most cases. One of these is known as the method of differences, or the **difference method**.

The following example illustrates the method of differences. Although the working appears initially to be going nowhere, it will eventually lead to a direct proof that

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

During the proof it will be assumed that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$, which has been established previously as the sum of an arithmetic series.

Consider the following identity:

$$n^3-(n-1)^3 \equiv n^3-\left(n^3-3n^2+3n-1\right)$$

i.e. $$n^3-(n-1)^3 \equiv 3n^2-3n+1. \qquad (1)$$

From this, the similar identity

$$(n-1)^3-(n-2)^3 \equiv 3(n-1)^2-3(n-1)+1$$

can be deduced by replacing n by $n-1$.

Also, replacing n by $n-2$ in equation (1) gives

$$(n-2)^3-(n-3)^3 \equiv 3(n-2)^2-3(n-2)+1, \text{ etc.}$$

Writing this sequence of results in a column form:

$$\begin{aligned}
n^3-(n-1)^3 &= 3n^2-3n+1\\
(n-1)^3-(n-2)^3 &= 3(n-1)^2-3(n-1)+1\\
(n-2)^3-(n-3)^3 &= 3(n-2)^2-3(n-2)+1\\
(n-3)^3-(n-4)^3 &= 3(n-3)^2-3(n-3)+1\\
\vdots \quad\quad \vdots \quad &\quad\quad \vdots \quad\quad\quad \vdots\\
3^3-2^3 &= 3(3)^2-3(3)+1\\
2^3-1^3 &= 3(2)^2-3(2)+1\\
1^3-0^3 &= 3(1)^2-3(1)+1
\end{aligned}$$

Now add up all the terms on the LHS: you get n^3-0^3, since all other terms appear once positively and once negatively, cancelling out. Adding up the RHS in three columns gives

$$3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + n$$

Thus $$n^3 = 3\sum_{r=1}^{n} r^2 - 3\sum_{r=1}^{n} r + n \qquad \left[\text{using } \sum_{r=1}^{n} r = \frac{1}{2}n(n+1)\right]$$

$$\Rightarrow \quad 3\sum_{r=1}^{n} r^2 = \left(n^3-n\right) + 3\times\frac{1}{2}n(n+1)$$

$$= n\left(n^2-1\right)+\frac{3}{2}n(n+1)$$

$$= \frac{2n}{2}(n-1)(n+1)+\frac{3}{2}n(n+1)$$

$$= \frac{n(n+1)}{2}\{2(n-1)+3\}$$

$$= \frac{1}{2}n(n+1)(2n+1)$$

Finally, dividing by 3 gives the required result,

$$\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$$

Example

By considering $n^5-(n-1)^5$ and similar expressions, find the formula for $\sum_{r=1}^{n} r^4$ in terms of n, assuming the results for $\sum_{r=1}^{n} r$, $\sum_{r=1}^{n} r^2$ and $\sum_{r=1}^{n} r^3$.

Solution

$$n^5-(n-1)^5 = 5n^4-10n^3+10n^2-5n+1$$

$$(n-1)^5-(n-2)^5 = 5(n-1)^4-10(n-1)^3+10(n-1)^2-5(n-1)+1$$

$$(n-2)^5-(n-3)^5 = 5(n-2)^4-10(n-2)^3+10(n-2)^2-5(n-2)+1$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$3^5-2^5 = 5(3)^4-10(3)^3+10(3)^2-5(3)+1$$

$$2^5-1^5 = 5(2)^4-10(2)^3+10(2)^2-5(2)+1$$

$$1^5-0^5 = 5(1)^4-10(1)^3+10(1)^2-5(1)+1$$

Adding: $\quad n^5-0^5 = 5\sum_{r=1}^{n} r^4 - 10\sum_{r=1}^{n} r^3 + 10\sum_{r=1}^{n} r^2 - 5\sum_{r=1}^{n} r + n$

Rearranging, and using the three standard results:

$$5\sum_{r=1}^{n} r^4 = \left(n^5 - n\right) + 10\,\frac{n^2}{4}(n+1)^2 - 10\,\frac{n}{6}(n+1)(2n+1) + 5\,\frac{n}{2}(n+1)$$

$$= \frac{6}{6}n(n-1)(n+1)\left(n^2+1\right) + \frac{15}{6}n^2(n+1)^2 - 10\,\frac{n}{6}(n+1)(2n+1) + \frac{15}{6}n(n+1)$$

$$= \frac{1}{6}n(n+1)\left\{6(n-1)\left(n^2+1\right) + 15n(n+1) - 10(2n+1) + 15\right\}$$

$$= \frac{1}{6}n(n+1)\left\{6n^3 - 6n^2 + 6n - 6 + 15n^2 + 15n - 20n - 10 + 15\right\}$$

$$= \frac{1}{6}n(n+1)\left(6n^3 + 9n^2 + n - 1\right)$$

$$= \frac{1}{6}n(n+1)(2n+1)\left(3n^2 + 3n - 1\right)$$

so that

$$\sum_{r=1}^{n} r^4 = \frac{1}{30}n(n+1)(2n+1)\left(3n^2+3n-1\right),$$

admittedly not a very appealing result! You will note, however, that the method of differences consists of adding two sides of a set of identities of a common form, where one side is expressed as a difference of successive terms. In this way, almost all terms cancel on this side in the summation.

The difference method can be employed in other circumstances also.

Example

Prove that

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = 1 - \frac{1}{(n+1)!}$$

Solution

Some ingenuity is needed here to turn the one term in the summation on the LHS into two terms. Note that

$$\frac{r}{(r+1)!} = \frac{r+1-1}{(r+1)!} = \frac{r+1}{(r+1)!} - \frac{1}{(r+1)!} = \frac{1}{r!} - \frac{1}{(r+1)!}$$

so that

$$\sum_{r=1}^{n} \frac{r}{(r+1)!} = \sum_{r=1}^{n}\left(\frac{1}{r!} - \frac{1}{(r+1)!}\right)$$

$$= \left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \dots$$

$$+ \left(\frac{1}{n-1!} - \frac{1}{n!}\right) + \left(\frac{1}{n!} - \frac{1}{(n+1)!}\right)$$

$$= \frac{1}{1!} - \frac{1}{(n+1)!} \qquad \text{[since all other terms cancel]}$$

$$= 1 - \frac{1}{(n+1)!}$$

The difference method can be summarised by the following, which has been described as the fundamental theorem of summation:

$$\boxed{\sum_{r=1}^{n} \{ f(r) - f(r-1) \} = f(n) - f(0)}$$

where f is any function suitably defined on the non-negative integers.

As an example take $f(r) = \sin\left(ar + \frac{a}{2}\right)$ for some constant a (not equal to a multiple of 2π). Then

$$\sin\left(ar + \frac{a}{2}\right) - \sin\left(a[r-1] + \frac{a}{2}\right)$$

$$= \sin\left(ar + \frac{a}{2}\right) - \sin\left(ar - \frac{a}{2}\right)$$

$$= 2\cos\left(\frac{ar + \frac{a}{2} + ar - \frac{a}{2}}{2}\right)\sin\left(\frac{ar + \frac{a}{2} - ar - \frac{a}{2}}{2}\right)$$

$$= 2\cos(ar)\sin\frac{a}{2}$$

and

$$\sum_{r=1}^{n} 2\cos(ar)\sin\frac{a}{2} = f(n) - f(0) = \sin\left(an + \frac{a}{2}\right) - \sin\left(\frac{a}{2}\right)$$

by the difference method; leading to the result

$$\sum_{r=1}^{n} \cos(ar) = \frac{1}{2}\left\{\frac{\sin\left(an + \frac{a}{2}\right)}{\sin\frac{a}{2}} - 1\right\}$$

or, since

$$\sin\left(an + \frac{a}{2}\right) - \sin\left(\frac{a}{2}\right) = 2\cos\left[\frac{a(n+1)}{2}\right]\sin\left(\frac{an}{2}\right),$$

$$\sum_{r=1}^{n}\cos(ar) = \frac{\cos\left[\frac{a(n+1)}{2}\right]\sin\left(\frac{an}{2}\right)}{\sin\frac{a}{2}}$$

If you find this approach a little confusing, it only enables you to save a line or two of working. In tricky cases, resorting to writing out the series in question and seeing the terms which cancel will prove much simpler to handle correctly.

Example

Express $\frac{1}{x(x+1)}$ in terms of partial fractions.

Hence show that $\sum_{r=1}^{n}\frac{1}{r(r+1)} = \frac{n}{n+1}$ for all positive integers n.

Solution

Assume $\frac{1}{x(x+1)} \equiv \frac{A}{x} + \frac{B}{x+1}$.

By the cover–up method (or multiplying through by $x(x+1)$ and comparing coefficients or substituting values), $A=1$ and $B=-1$.

Then
$$\sum_{r=1}^{n}\frac{1}{r(r+1)} = \sum_{r=1}^{n}\left(\frac{1}{r} - \frac{1}{r+1}\right)$$

$$= \left(\frac{1}{1}-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\dots$$

$$+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$= \frac{1}{1}-\frac{1}{n+1} = \frac{n+1}{n+1}-\frac{1}{n+1}$$

$$= \frac{n}{n+1}, \text{ as required.}$$

Alternatively, note that this is $\sum_{r=1}^{n}\left\{f(r)-f(r-1)\right\}$ with

$f(r) = -\frac{1}{r+1}$. The sum is then

$$f(n)-f(0) = -\frac{1}{n+1} - \left(-\frac{1}{0+1}\right) = 1 - \frac{1}{n+1}, \text{ etc. as before.}$$

Exercise 6C

1. Show that $3r(r+1) = r(r+1)(r+2) - r(r-1)(r+1)$ and deduce that $\sum_{r=1}^{n} r(r+1) = \frac{1}{3}n(n+1)(n+2)$.

2. Express $\frac{1}{r(r+2)}$ in partial fractions, and hence show that $\sum_{r=1}^{n} \frac{1}{r(r+2)} = \frac{3}{4} - \frac{2n+3}{2(n+1)(n+2)}$

3. Simplify the expression $(r+1)! - r!$ and hence prove that $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$.

4. Simplify the expression $\left(\sqrt{r} + \sqrt{r-1}\right)\left(\sqrt{r} - \sqrt{r-1}\right)$.

   Hence prove that $\sum_{r=1}^{n} \frac{1}{\sqrt{r} + \sqrt{r-1}} = \sqrt{n}$ for all positive integers n.

5. Given that r is a positive integer and $f(r) = \frac{1}{r^2}$, find a single expression for $f(r) - f(r+1)$. Hence prove that $\sum_{r=1}^{4n} \left(\frac{2r+1}{r^2(r+1)^2} \right) = \frac{(3n+1)(5n+1)}{n^2(4n+1)^2}$

6. Prove the identity
   $$\frac{2r+3}{r(r+1)} - \frac{2r+5}{(r+1)(r+2)} \equiv \frac{2(r+3)}{r(r+1)(r+2)}.$$
   Hence, or otherwise, find the sum of the series
   $$S_n = \frac{8}{1\times2\times3} + \frac{10}{2\times3\times4} + \ldots + \frac{2(n+3)}{n(n+1)(n+2)}$$

7. Use the method of differences to find $\sum_{r=1}^{n} \frac{1}{(n+r-1)(n+r)}$ in terms of n.

8. Prove that $\sum_{r=1}^{n} \frac{r^2+r+1}{r^2+r} = n+1-\frac{1}{n+1}$.

9. Prove the identity
   $$\cos(A-B) - \cos(A+B) = 2\sin A \sin B.$$
   Hence prove that
   $$\sin\theta\left\{\sin\theta + \sin3\theta + \sin5\theta + \ldots + \sin(2n-1)\theta\right\} = \sin^2 n\theta$$

*10. Express $\frac{1}{r(r+1)(r+2)}$ in partial fractions. Hence prove the result
   $$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

*11. By considering $\frac{1}{1+a^{n-1}} - \frac{1}{1+a^n}$, show that
   $$\sum_{r=1}^{N} \frac{a^{n-1}}{\left(1+a^{n-1}\right)\left(1+a^n\right)} = \frac{a^N - 1}{2(a-1)\left(a^N+1\right)},$$
   where a is positive and $a \neq 1$. Deduce that
   $$\sum_{r=1}^{N} \frac{2^n}{\left(1+2^{n-1}\right)\left(1+2^n\right)} < 1. \qquad \text{(Cambridge)}$$

*12. Use the difference method to show that
   $$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}} = 1 - \frac{1}{\sqrt{n+1}}$$

6.5 Infinite series

Thus far in this chapter you have studied only finite series. You have, however, already encountered at least one example of a series which can be 'summed-to-infinity' without simply obtaining an 'infinitely large number', namely the geometric series

$$a + ar + ar^2 + \ldots + ar^{n-1} + \ldots,$$

provided that $-1 < r < 1$.

The sum to n terms of a geometric series can be expressed as

$$S_n = \frac{a\left(1-r^n\right)}{1-r} \qquad (r \neq 1)$$

which can be split up as

$$\frac{a}{1-r}-\frac{ar^n}{1-r}.$$

For $|r|<1$, $r^n \to 0$ as $n \to \infty$ (said 'r^n tends to zero as n tends to infinity'), in which case the second term $\to 0$ also. This is written

$$\lim_{n \to \infty}\left(\frac{ar^n}{1-r}\right)=0$$

and said 'the limit as n tends to infinity of $\left(\frac{ar^n}{1-r}\right)$ is zero'. The infinite geometric series converges in this case to the number

$$\lim_{n \to \infty}\{S_n\}=\frac{a}{1-r},$$

or $\qquad S_\infty=\frac{a}{1-r}$ for short.

When $|r|>1$, $\frac{ar^n}{1-r} \not\to 0$, and the geometric series 'diverges', having no limit.

The sum-to-infinity of any convergent series is defined in the following way: given S_n, the sum to n terms of a series,

$$S_\infty=\lim_{n\to\infty}(S_n)$$

Example

From an earlier example, $\sum_{r=1}^{n}\frac{r}{(r+1)!}=1-\frac{1}{(n+1)!}=S_n$.

Then

$$S_\infty=\lim_{n\to\infty}\left(1-\frac{1}{(n+1)!}\right)=1$$

since $\qquad \frac{1}{(n+1)!}\to 0$ as $n\to\infty$;

i.e. the sum of the infinite series

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\dots \text{ is } 1.$$

Example

Given that $S_n = \dfrac{2n^2 + 3n + 1}{3n^2 + 5n - 7}$, find $\lim_{n\to\infty}(S_n)$.

Solution

By dividing the numerator and denominator of S_n by n^2, S_n can be written in the form

$$S_n = \frac{\left(2 + \frac{3}{n} + \frac{1}{n^2}\right)}{\left(3 + \frac{5}{n} - \frac{7}{n^2}\right)}$$

Now, as $n \to \infty$, $\dfrac{1}{n} \to 0$ and $\dfrac{1}{n^2} \to 0$ so that $S_n \to \dfrac{2}{3}$.

Thus $S_\infty = \dfrac{2}{3}$.

Another way of deducing this is to observe that, for n 'large', the numerator is $\approx 2n^2$; the '$3n$' and the '1' paling into insignificance in comparison; while the denominator is $\approx 3n^2$ for similar reasons. Then $S_n \approx \dfrac{2n^2}{3n^2} = \dfrac{2}{3}$.

Exercise 6D

1. The first term of a geometric series is 8 and the sum to infinity is 400. Find the common ratio.

2. The first, second and third terms of a geometric series are p, p^2 and q respectively, where $p < 0$. The first, second and third terms of an arithmetic series are p, q, p^2 respectively.

   (a) Show that $p = -\frac{1}{2}$ and find the value of q.

   (b) Find the sum-to-infinity of the geometric series.

3. Given S_n, deduce the value of S_∞ in each of the following cases:

   (a) $S_n = \dfrac{n}{2n+1}$

   (b) $S_n = \dfrac{1}{4} - \dfrac{1}{2(n+1)(n+2)}$

   (c) $S_n = \dfrac{(3n+1)(5n+1)}{(4n+1)^2}$

   (d) $S_n = \dfrac{(2n+3)(n+1)}{n(n+2)(n+4)}$

   (e) $S_n = \dfrac{n(2n+5)}{(n+1)(n+2)}$

   (f) $S_n = \dfrac{5n+11}{2(n+1)(n+2)}$

   (g) $S_n = \dfrac{3}{4} - \dfrac{2n+3}{2(n+1)(n+2)}$

4. Given $S(n) = n \times 2^{n+1}$, find $\lim_{n\to\infty}\left(\dfrac{S(n+1)}{S(n)}\right)$.

5. Assuming the results $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$ and $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$, find an expression for $\sum_{r=1}^{n} r(r+1)$ in terms of n. Hence determine

$$\lim_{n\to\infty} \frac{\sum_{r=1}^{n} r(r+1)}{\left(\sum_{r=1}^{n} r\right)^{\frac{3}{2}}}$$

6. A geometric series is given by

$$e^{3x} + 3e^{x} + 9e^{-x} + \ldots$$

(a) Find the value of the sum-to-infinity in the case when $x = \ln 2$.

(b) Determine the ranges of values of x for which a sum-to-infinity exists.

6.6 Infinite binomial series

In *Pure Mathematics* you will have encountered the **binomial theorem** for positive integers n; that is, the (finite) series expansions for $(a+b)^n$ in terms of a, b and n when $n = 1, 2, 3, \ldots$

Remember, $(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{n}b^n$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r,$$

where $$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} \qquad (\text{for } 0 \le r \le n)$$

are called the **binomial coefficients**.

From this, the result

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \qquad (2)$$

can be deduced, and this series terminates (i.e. is finite) in the cases when n is a positive integer, since the factor $(n-n)$ appears in all the coefficients of powers of x from x^{n+1} onwards.

What happens: when n is not a positive integer?

when n is a negative integer such as $n = -3$?

when n is a rational such as $\frac{1}{2}$?

In these cases, the factor of zero no longer appears at any stage in the expansion and the series continues indefinitely. Although the binomial coefficient $\binom{n}{r}$ or ${}^{n}C_{r}$ is only defined for non-negative integers n and r, (note that $\binom{n}{r}$ is defined to be zero when $r > n$), the coefficients still take the form given in equation (2). The proof of this result forms part of an activity later in the chapter.

Example

Expand $(1+x)^{-3}$ in ascending powers of x, up to and including the term in x^3.

Solution

Using the given expansion in the form

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

gives

$$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-4)}{2}x^2 + \frac{(-3)(-4)(-5)}{6}x^3 + \ldots$$

$$= 1 - 3x + 6x^2 - 10x^3 + \ldots$$

Example

Expand $(1-6x)^{\frac{1}{2}}$ up to the term in x^3.

Solution

$$(1-6x)^{\frac{1}{2}} = \left(1+[-6x]\right)^{\frac{1}{2}}$$

$$= 1 + \tfrac{1}{2}(-6x) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-6x)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-6x)^3 + \ldots$$

'x' is now $-6x$

$$(-6x)^2 = (-6x)(-6x) = 36x^2$$

$$= 1 - 3x - \frac{9}{2}x^2 - \frac{27}{2}x^3 \ldots$$

Example

Expand $(4+3x)^{-\frac{1}{2}}$ up to the term in x^3.

Solution

Now the form of the general binomial expansion given is for (1+ 'something')$^{\text{power}}$, while this example is (4 + 'something')$^{\text{power}}$.

Although it is quite possible to adapt back to the form $(a+b)^n$, you are strongly advised to begin each binomial expansion (of the infinite variety) with (1 + ...)

This is easily done here in the following way:

$$4+3x=4\left(1+\frac{3}{4}x\right)$$

so that
$$(4+3x)^{-\frac{1}{2}}=\left[4\left(1+\frac{3}{4}x\right)\right]^{-\frac{1}{2}}$$

$$=4^{-\frac{1}{2}}\left(1+\frac{3}{4}x\right)^{-\frac{1}{2}} \qquad [\textit{Not } 4\left(1+\frac{3}{4}x\right)^{-\frac{1}{2}}]$$

$$=\frac{1}{2}\left(1+\frac{3}{4}x\right)^{-\frac{1}{2}}$$

To demonstrate this simple algebraic technique has taken four lines of working, as this is an example. It can be performed automatically in your working provided you do not make the mistake highlighted in the bracket above, which is a very common error indeed.

To continue,

$$(4+3x)^{-\frac{1}{2}}=\frac{1}{2}\left(1+\frac{3}{4}x\right)^{-\frac{1}{2}}$$

$$=\frac{1}{2}\left\{1+\left(-\frac{1}{2}\right)\left(\frac{3}{4}x\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{3}{4}x\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{3}{4}x\right)^3+\dots\right\}$$

$$=\frac{1}{2}\left\{1-\frac{3}{8}x+\frac{27}{128}x^2-\frac{135}{1024}x^3+\dots\right\}$$

$$=\frac{1}{2}-\frac{3}{16}x+\frac{27}{256}x^2-\frac{135}{2048}x^3+\dots$$

6.7 Conditions for convergence of the binomial series

The infinite geometric series

$$1 + x + x^2 + x^3 + \ldots + x^n + \ldots$$

has $$S_\infty = \frac{1}{1-x} = (1-x)^{-1}.$$

This familiar result is therefore an example of an infinite binomial expansion, and you have already seen that this series **converges** only if $-1 < x < 1$.

When n is a positive integer, or zero, the binomial series for $(1+x)^n$ converges for all values of x, since the expansion is finite. This is not the case, however, when n is not a positive integer, as already noted.

*Activity 2

By writing $(1+x)^n$ as $\sum_{r=0}^{\infty} u_r$, with

$$u_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\times 2\times 3\ldots r}x^r,$$

the sequence $\{u_r\}$ can be defined by

$$u_1 = 1 \text{ and } u_r = \frac{(n-r+1)}{r}x \times u_{r-1} \text{for } r \geq 2.$$

Write a program to find the partial sums of $(1+X)^N$ as follows

(i) Choose suitable input values of N and X.

(ii) Define variables S, M, R and U with initial values $S=1,\ M=N,\ R=0,\ U=1$. (S is the sum so far; M will be the additional factor of $(N-R+1)$ at each stage; R is the number of terms added after the 1; and U is the last term added to the sum.)

(iii) Repeat the steps

Print R and S at this point

Now increase R by 1;

U becomes $\dfrac{(U\times M\times X)}{R}$;

S becomes $S+U$;

decrease M by 1.

This could be done on a programmable calculator. You need to be able to repeat the step once each time you press (for instance) the 'EXECUTE' button.

The purpose of the investigation is to choose values of N and X and discover which values of X lead to the convergence of $(1+X)^N$ for the various possible values of N. The convergence/ divergence of the geometric series suggests the investigation of the cases

$$X<-1,\ X=-1,\ -1<X<1,\ X=1,\ X>1$$

for various possible values of N.

The conclusions you should have reached from the Activity above are as follows.

When n is not a positive integer (or zero), the series expansion of $(1+x)^n$ is convergent

(i) for $-1<x<1$ when $n\le -1$,

(ii) for $-1<x\le 1$ when $-1<n<0$,

(iii) for $-1\le x\le 1$ when $n>0$.

The above results are not widely acknowledged and you are not required to know them in this detail. However, you will be expected to be able to quote the following, simplified rule:

> For n not a positive integer, or zero, the series expansion of $(1+x)^n$ is convergent for $-1<x<1$ in general.

Example

State the values of x for which the following series expansions converge:

(a) $(2-x)^{-7}$ (b) $(1+4x)^{\frac{3}{4}}$ (c) $\dfrac{1}{\sqrt[3]{3+2x}}$

Solution

(a) $(2-x)^{-7}=2^{-7}\left(1-\dfrac{x}{2}\right)^{-7}$, which converges for $-1<-\dfrac{x}{2}<1$,

i.e. for $-2<x<2$.

(b) $(1+4x)^{\frac{3}{4}}$ converges for $-1<4x<1$, i.e. for $-\dfrac{1}{4}<x<\dfrac{1}{4}$

(In fact $-\dfrac{1}{4}\le x\le \dfrac{1}{4}$)

(c) $\frac{1}{\sqrt[3]{3+2x}} = (3+2x)^{-\frac{1}{3}} = 3^{-\frac{1}{3}}\left(1+\frac{2x}{3}\right)^{-\frac{1}{3}}$, converging for

$-1 < \frac{2x}{3} < 1 \Rightarrow -\frac{3}{2} < x < \frac{3}{2}$. (In fact, $-\frac{3}{2} < x \le \frac{3}{2}$)

Example

Write down the expansion of $(1-2x)^{\frac{1}{2}}$ up to and including the term in x^3. By setting $x = \frac{1}{100}$, use this expansion to find an approximation to $\sqrt{2}$ to eight places of decimals.

Solution

$$(1-2x)^{\frac{1}{2}} = 1+\frac{1}{2}(-2x)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(-2x)^2+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(-2x)^3+\ldots$$

$$= 1-x-\frac{1}{2}x^2-\frac{1}{2}x^3\ldots$$

Setting $x = \frac{1}{100} \Rightarrow (1-0.02)^{\frac{1}{2}} = 1-\frac{1}{100}-\frac{1}{2}\left(\frac{1}{100}\right)^2-\frac{1}{2}\left(\frac{1}{100}\right)^3\ldots$

$$\Rightarrow \sqrt{0.98} = 1-0.01-0.000\,05-0.000\,000\,5\ldots$$

Now $0.98 = \frac{98}{100} = \frac{49}{100}\times 2$ so that

$$\sqrt{0.98} = \frac{7}{10}\sqrt{2}$$

Then $\frac{7}{10}\sqrt{2} \approx 0.989\,949\,5$

and $\sqrt{2} \approx \frac{10\times 0.9899495}{7} = 1.414\,213\,57$ (to 8 d.p.)

All this working can easily be done without a calculator, and to 8 d.p. $\sqrt{2} = 1.414\,213\,56$.

Example

Given that $-1 < x < 1$, find the expansion of $\frac{3-2x}{(1+x)\left(4+x^2\right)}$ in ascending powers of x, up to and including the term in x^3.

Solution

$$\frac{3-2x}{(1+x)\left(4+x^2\right)}=(3-2x)(1+x)^{-1}\left(4+x^2\right)^{-1}$$

$$=(3-2x)(1+x)^{-1}\frac{1}{4}\left(1+\frac{x^2}{4}\right)^{-1}$$

$$=\frac{1}{4}(3-2x)\left(1-x+x^2-x^3+\ldots\right)\left(1-\frac{x^2}{4}+\ldots\right)$$

$$=\frac{1}{4}(3-2x)\left(1-x+\frac{3}{4}x^2-\frac{3}{4}x^3\ldots\right)$$

$$=\frac{3}{4}-\frac{5}{4}x+\frac{17}{16}x^2-\frac{15}{16}x^3\ldots$$

Alternatively, you could write $\frac{3-2x}{(1+x)\left(4+x^2\right)}$ in partial fractions:

$$\frac{1}{1+x}-\frac{x+1}{4+x^2}=(1+x)^{-1}-(1+x)\frac{1}{4}\left(1+\frac{x^2}{4}\right)^{-1}$$ etc., as before.

Exercise 6E

1. Write down the first four terms in the binomial expansions of

   (a) $(1+3x)^{\frac{1}{3}}$ (b) $(2+x)^{-1}$ (c) $\sqrt{1+\frac{x}{2}}$

   (d) $(5-3x)^{-2}$ (e) $\frac{1+x}{1-x}$ (f) $\frac{1-x}{\sqrt{1+x^2}}$

   For each part above, state the (simplified) range of values of x for which the expansion is valid.

2. Find, in simplified form, the first three non-zero terms in ascending powers of x of the series expansions of

   (a) $\frac{1+3x}{(1-2x)^4}$ for $|x|<\frac{1}{2}$

   (b) $(2+x)\sqrt{1-x}$ for $|x|<1$

3. Expand $\frac{3+x}{\left(1+x^2\right)(1+2x)}$ in ascending powers of x, up to and including the term in x^3.

   State the range of values of x for which this expansion is valid.

4. Obtain the expansion of $(16+y)^{\frac{1}{2}}$ in ascending powers of y up to and including the term in y^2. Hence show that if k^3 and higher powers of k are neglected, $\sqrt{16+4k+k^2}\approx 4+\frac{k}{2}+\frac{3k^2}{32}$.

5. Obtain the expansion in ascending powers of x, up to and including the term in x^3, of

   $$\frac{1+5x}{(1+2x)^{\frac{1}{2}}} \text{ for } |x|<\frac{1}{2}.$$

   By putting $x=0.04$ deduce an approximate value of $\frac{1}{\sqrt{3}}$, giving your answer to three decimal places.

6. Expand $\dfrac{x}{(1-2x)^2(1-3x)}$ up to the term in x^3.

   For what range of values of x is the expansion valid?

7. Write down the binomial series for $\sqrt{\dfrac{1-x^2}{1+x^2}}$ up to and including the term in x^4.

8. Determine the series expansion of $(1-x)^{-\frac{1}{2}}$ up to and including the term in x^3. By setting $x=\dfrac{1}{10}$ determine, without the aid of a calculator, an approximate value of $\sqrt{10}$, giving your answer to 7 decimal places.

9. Use the binomial theorem to show that

$$\frac{x^2}{\sqrt{4-x^2}}=\frac{1}{2}x^2+\frac{1}{16}x^4+kx^6+\ldots\left(|x|<2\right)$$

   for some constant k, and state its value.

   Hence show by integrating these first three terms of the series, that the value of the integral

$$I=\int_0^1\frac{x^2dx}{\sqrt{4-x^2}}$$

   is approximately 0.1808.

6.8 Maclaurin expansions

Along with infinite geometric series and the general binomial series, many other functions can also be represented as **power series** (which can be thought of as polynomials of infinite degree). For such a power series to exist, the function in question needs to be infinitely differentiable, with each derivative, $f^{(n)}(x) \quad (n=1, 2, 3, \ldots)$, capable of being evaluated at $x=0$ and in some range of values of x containing 0. Consider, for example, $f(x)=e^x$. One way of defining this exponential function is by choosing the base number e in such a way that $f'(x)=e^x$ also; that is, the function is its own derivative. Hence $f^{(n)}(x)=e^x$ for $n=1,2,3,\ldots$. Maclaurin's theorem can now be illustrated in this case.

Writing $e^x=a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+\ldots$, where a_r is the coefficient of x^r for $0\le r<\infty$, repeated differentiation gives

$$e^x=a_1+2a_2x+3a_3x^2+4a_4x^3+\ldots$$

and $$e^x=2a_2+3\times2\times a_3x+4\times3a_4x^2+\ldots$$

and $$e^x=3\times2a_3+4\times3\times2a_4x+\ldots$$

and $$e^x=4\times3\times2a_4+\ldots$$ etc.

Since $e^0=1$, substituting $x=0$ in each line gives

$$1=a_0,\quad 1=a_1,\quad 1=2a_2,\quad 1=3\times2a_3,\quad 1=4\times3\times2a_4\ \ldots$$

In general, it is easily seen that the nth derivative evaluated at $x = 0$, $f^{(n)}(0)$, gives

$$1 = n(n-1)(n-2)\dots 3\times 2a_n,$$

so that $$a_n = \frac{1}{n!}$$

The power series for e^x is then

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$= \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

and this is called the Maclaurin series, or Maclaurin expansion, of e^x.

How can you find the series for e^{-x}?

To obtain the power series for e^{-x} it is not necessary to repeat the above process: simply replacing x by $-x$ in the series for e^x gives

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!}$$

(Note that $(-1)^r = -1$ when r is odd, and $(-1)^r = +1$ when r is even.)

For a general function $f(x)$, satisfying the relevant conditions, Maclaurin's theorem gives the expansion

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots$$

i.e. $$f(x) = \sum_{r=0}^{\infty} f^{(r)}(0)\frac{x^r}{r!}$$

Activity 3

Using the results $\dfrac{d}{dx}(\sin x) = \cos x$ and $\dfrac{d}{dx}(\cos x) = -\sin x$, find the Maclaurin series for $\sin x$. Without repeating the whole process, deduce the series for $\cos x$.

Activity 4

Use Maclaurin's theorem to prove that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

Example

Determine the Maclaurin series for $\tanh x$ up to and including the term in x^3.

Solution

$$f(x) = \tanh x$$

$$f'(x) = \operatorname{sech}^2 x$$

$$f''(x) = 2\operatorname{sech} x(-\operatorname{sech} x \tanh x)$$

$$= -2\operatorname{sech}^2 x \tanh x$$

$$f'''(x) = -2\operatorname{sech}^2 x \times \operatorname{sech}^2 x - 2\tanh x\left(-2\operatorname{sech}^2 x \tanh x\right)$$

$$= -2\operatorname{sech}^4 x + 4\operatorname{sech}^2 x \tanh^2 x$$

This gives

$$f(0) = 0 \qquad (\text{since } \tanh 0 = 0)$$

$$f'(0) = 1 \qquad (\text{since } \operatorname{sech} 0 = 1)$$

$$f''(0) = -2 \times 1^2 \times 0 = 0$$

$$f'''(0) = -2 \times 1^4 + 4 \times 1^2 \times 0^2 = -2$$

So $\quad \tanh x = 0 + 1 \times x + 0 \times \dfrac{x^2}{2} + (-2)\dfrac{x^3}{6} + \ldots$

$$= x - \frac{x^3}{3} + \ldots$$

The following alternative approach is quite useful.

Write $y = \tanh x$. Then $\frac{dy}{dx} = \text{sech}^2 x = 1 - \tanh^2 x = 1 - y^2$. Next, differentiate $\left\{\frac{dy}{dx} = 1 - y^2\right\}$ with respect to x implicitly, to give

$$\frac{d^2y}{dx^2} = -2y\frac{dy}{dx}$$

and

$$\frac{d^3y}{dx^3} = -2y\left(\frac{d^2y}{dx^2}\right) - 2\frac{dy}{dx}\left(\frac{dy}{dx}\right) \quad \text{(using the Product Rule)}$$

$$= -2y\frac{d^2y}{dx^2} - 2\left(\frac{dy}{dx}\right)^2 \qquad \text{etc.}$$

Then, when $x = 0, y = 0$, $\frac{dy}{dx} = 1 - 0^2 = 1$, $\frac{d^2y}{dx^2} = -2 \times 0 \times 1 = 0$, $\frac{d^3y}{dx^3} = -2 \times 0 \times 0 - 2 \times 1^2 = -2, \ldots.$

Activity 5

1. Determine the Maclaurin expansions of the following functions, up to and including the term in x^3:

   (a) $\frac{1}{1-x}$

   (b) $\ln(1+x)$. (Why is it not possible to find a power series for $\ln x$?)

   (c) $\ln(1-x)$.

   Describe how the series for (c) can be found
   (i) using the answer to (b);
   (ii) using the answer to (a).

2. Use the results $\frac{d}{dx}(\sinh x) = \cosh x$ and $\frac{d}{dx}(\cosh x) = \sinh x$ to find the Maclaurin series for $\sinh x$ and $\cosh x$.

3. Prove Euler's relationship $e^{i\theta} = \cos\theta + i\sin\theta$.

4. Use the Maclaurin series for $\sin x, \cos x, \sinh x$ and $\cosh x$ to show that $\sin ix = i\sinh x$ and $\cos ix = \cosh x$, where $i^2 = -1$.

Example

Given that the first non-zero term in the series expansion of $e^{-px} - (1+2x)^{-q}$ in ascending powers of x is $-4x^2$, find the value of p and the value of q.

Solution

$$e^{-px} = 1 - (px) + \frac{(px)^2}{2!} - \ldots$$
$$= 1 - px + \frac{1}{2}p^2x^2 + \ldots$$

and $$(1+2x)^{-q} = 1 + (-q)(2x) + \frac{(-q)(-q-1)}{2!}(2x)^2 + \ldots$$

$$= 1 - 2qx + 2q(q+1)\,x^2 + \ldots$$

Subtracting

$$e^{-px} - (1+2x)^{-q} = (2q-p)x + \left(\frac{1}{2}p^2 - 2q^2 - 2q\right)x^2 + \ldots$$

It is given that the coefficient of x is zero. Thus $p = 2q$. The coefficient of x^2 is then $\frac{1}{2}(2q)^2 - 2q^2 - 2q$. As this is equal to -4, it follows that $q = 2$ and $p = 4$.

Validity

The series expansions for e^x, e^{-x}, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ (also for $\tan x$ and $\tanh x$, although these are not very straightforward and so cannot easily be remembered) are valid for all real x.

However, the series for $\ln(1+x)$ is valid for $-1 < x \leq 1$ only, while the series for $\ln(1-x)$ is valid for $-1 \leq x < 1$.

It is not within the scope of this course formally to establish these results, but you should learn them and you may be expected to use them.

Exercise 6F

1. Write down the first four terms in the series expansions of

(a) e^{-2x} (b) $\ln\left(1+\frac{x}{2}\right)$ (c) $\sin 3x$

(d) $\cos\frac{1}{2}x$ (e) $\ln\left(\frac{1+x}{1-x}\right)$ (f) $\sin 2x-\cos 4x$

(g) $\frac{e^{-x}}{2+x}$ (h) $\left(1+e^{-x}\right)\left(1-2e^{-x}\right)$

2. Use Maclaurin's theorem to determine the series expansions of the following functions, up to and including the term in x^3:

(a) $f(x)=\sin^{-1}x$ (b) $f(x)=\sqrt{\cos x}$

(c) $f(x)=\tan^{-1}\left\{\sinh(x+\ln 2)\right\}$

(d) $f(x)=2^x$

(For (d): $\frac{d}{dx}\left(a^x\right)=\frac{d}{dx}\left(e^{x\ln a}\right)=\ln a.e^{x\ln a}=a^x\ln a$)

3. The series expansion, in ascending powers of θ, of $4\cos 2\theta-\cos 4\theta$ begins $A+B\theta^4$, where A and B are integers. Find the values of A and B.

4. Use the Maclaurin series for $\cos x$ and $\ln(1+y)$ to show that $\ln(\cos x)=Ax^2+Bx^4$ when terms in x^5 and higher powers of x can be neglected. State the values of the constants A and B.

5. Given that $y=\tan x$, show that $\frac{dy}{dx}=1+y^2$ and $\frac{d^2y}{dx^2}=2y\left(1+y^2\right)$. Obtain an expression for $\frac{d^3y}{dx^3}$ in terms of y. Evaluate these derivatives at $x=0$ and hence write down the Maclaurin series of $\tan x$, including the term in x^3.

6. Given that $0<x<1$, write down the sum of the infinite series $x+\frac{x^2}{2}+\frac{x^3}{3}+\ldots+\frac{x^r}{r}+\ldots$ By integrating this series term by term, show that

$$\frac{x^2}{1\times 2}+\frac{x^3}{2\times 3}+\frac{x^4}{3\times 4}+\ldots+\frac{x^{r+1}}{r(r+1)}+\ldots$$

$$=x+(1-x)\ln(1-x)$$

Hence, or otherwise, find the sum of the infinite series

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}\left(\frac{1}{2}\right)+\frac{1}{3\times 4}\left(\frac{1}{2}\right)^2+\ldots+\frac{1}{r(r+1)}\left(\frac{1}{2}\right)^{r-1}+\ldots$$

7. The function $y=f(x)$ satisfies the differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=36\sin 3x$, and is such that $f(0)=2$ and $f'(0)=-4$. Write down the value of $f''(0)$ and obtain the value of $f'''(0)$. Hence obtain the Maclaurin expansion of $f(x)$ in ascending powers of x up to and including the term in x^3.

8. Given $f(x)=\frac{3}{1+x}+\frac{1}{1-2x}+\frac{2}{(1-2x)^2}$, show that $\ln\left\{f(x)\right\}=\ln 6+Ax+Bx^2$, provided that terms in x^3 and higher terms of x can be neglected. Find the value of A and the value of B.

9. Write down the expansion of e^{2x} in ascending powers of x up to and including the term in x^3. Show that the series expansion of $\frac{e^{2x}}{1+x}$ is $1+x+x^2+\ldots$ $\left(|x|<1\right)$ and find the coefficient of x^3 in this expansion. Use these four terms of the expansion to find an approximation to $\int_0^{\frac{1}{4}}\frac{e^{2x}}{1+x}dx$, giving your answer to 3 decimal places.

10. Show that when θ is small enough for θ^2 and higher powers of θ to be neglected, $(2-\tan\theta)(1+\sin 2\theta)=2+3\theta$. Hence find an approximation for $\int_0^{0.02}(2-\tan\theta)(1+\sin 2\theta)d\theta$, giving your answer to 4 decimal places.

11. Given that $-\frac{\pi}{2}<x<\frac{\pi}{2}$ and that $y=(\sec x+\tan x)^{\frac{1}{2}}$, prove that

(a) $2\frac{dy}{dx}=y\sec x$, and

(b) $4\frac{d^2y}{dx^2}=y\sec x(\sec x+2\tan x)$.

Use these results to find the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x=0$. Find the value of $\frac{d^3y}{dx^3}$ when $x=0$. Find a cubic polynomial approximation for $(\sec x+\tan x)^{\frac{1}{2}}$ when x is small and hence deduce an estimate to 5 decimal places for the integral $\int_0^{0.1}(\sec x+\tan x)^{\frac{1}{2}}dx$.

12. Given that $y = \ln(1+\sin x)$, find $\frac{dy}{dx}$ and show that $(1+\sin x)\frac{d^2y}{dx^2} + 1 = 0$. Find the fourth-degree polynomial approximation to $y = \ln(1+\sin\theta)$.

Hence show that if x^5 and higher powers of x are neglected,

$$\ln\left(\frac{1+\sin x}{1+x}\right) = k\left(x^4 - x^3\right)$$

where k is a constant. State the value of k.

6.9 General terms

In some cases, you have already seen and worked with the general terms of a number of the series expansions encountered in this chapter. For instance, the general term of the binomial series for $(1+x)^n$ is

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

Note: '*n*' has a specific value here and should not therefore be used as a variable.

In most cases it will be the term in x^n that will be required. The series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

has general term $\frac{x^n}{n!}$, as has already been seen. With finite series, n was usually taken as the last term in the summation and r used as the summation index, but with infinite series there is no last term and n is often used. As long as you do not get confused, or use n as a variable when it is also being used as a fixed value at the same time, then there is no problem. The general term is the one which represents the form of each term of the series.

Thus $$\sum_{n=0}^{\infty}\frac{x^n}{n!} = e^x.$$

In some cases, such as the series for $\sin x$,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

only some powers of x are non-zero; here, the odd ones – the coefficient of x^n is zero when x is even. This is easily dealt with

$$\sin x = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n-1)!}$$

(Check that this gives the correct terms with the correct signs.)

Activity 6

Write down the general terms in the series expansions of

$$e^{-x}, \quad \cos x, \quad \ln(1+x), \quad \ln(1-x), \quad \frac{1}{1-x}, \quad \sinh x, \quad \cosh x$$

Example

Prove that the coefficient of x^k in the binomial expansion of

$\frac{1}{(1-x)^2}$ is $k+1$.

Method 1

Using the expansion $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots + x^k + x^{k+1} + \ldots,$
for which all the coefficients are 1 (remember this is the infinite geometric series), and differentiating:

$$\frac{1}{(1-x)^2} = 0 + 1 + 2x + 3x^2 + \ldots + kx^{k-1} + (k+1)x^k + \ldots$$

and the coefficient of x^k is seen to be $(k+1)$, as required.

Method 2

Using the general term of the binomial series,

$$\frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}x^k,$$

with $n = -2$ and $'x' = -x$, gives the general term

$$\frac{(-2)(-3)(-4)\ldots(-2-k+1)}{k!}(-x)^k$$

$$= (-1)^k \frac{2 \times 3 \times 4 \ldots (k+1)}{k!}(-1)^k x^k$$ (taking out the factor of –1 from each of the k terms in the numerator)

$$= (-1)^{2k} \frac{(k+1)!}{k!} x^k$$

$$= 1(k+1)x^k$$ (since $2k$ is even)

$$= (k+1)x^k$$

So that x^k has coefficient $(k+1)$.

Exercise 6G

1. Find the coefficient of x^n in the series expansions of

   (a) e^{3x} (b) $\ln(1+4x)$ (c) $\cos 2x$

   (d) $\sin^2 x$ (e) e^{1-x} (f) $\sinh \frac{x}{2}$

2. Express $f(x)=\frac{1}{(2-x)(1+x)}$ in partial fractions.

   Hence show that the coefficient of $x^n\ (n \geq 0)$ in the series expansion of $f(x)$ is $\frac{1}{3}\left\{\frac{1}{2^{n+1}}+(-1)^n\right\}$.

3. Find the values of the constants A, B and C for which $\frac{x}{(1-2x)^2(1-3x)} \equiv \frac{A}{1-3x}+\frac{B}{1-2x}+\frac{C}{(1-2x)^2}$.

   Hence write down the series expansion of $\frac{x}{(1-2x)^2(1-3x)}$ in ascending powers of x, up to and including the term in x^3. For what range of values of x is this expansion valid? Determine, in terms of n, the coefficient of x^n in this expansion.

4. Show that the coefficient of x^n in the binomial expansion of $(1-x)^{-4}$ is $\binom{n+3}{3}$.

5. Determine an expression for the coefficient of $x^n\ (n \geq 2)$ in the series expansion of $\frac{x^2+3}{x+1}$, where $|x|<1$.

6. Find the general term in the series expansion of $\frac{1+3x}{(1-2x)^4}$ for $|x|<\frac{1}{2}$.

7. For $n \geq 1$, prove that the coefficient of x^n in the power series expansion of $\frac{1+x+x^2}{(1-x)^2}$ is $3n$.

8. Given that, $f(x)=\ln\left\{\frac{(1-3x)^2}{1+2x}\right\}$

   (a) determine the Maclaurin series of $f(x)$ up to and including the term in x^3;

   (b) state the range of values of x for which the series in (a) is valid;

   (c) find the coefficient of x^n in this expansion.

6.10 Miscellaneous Exercises

1. Mr Brown invests £350 a year at the beginning of every year in a savings scheme. At the end of each year, interest of 7% of the total so far invested is added to the scheme. Show that, if no money is withdrawn, there will be £S in the scheme at the end of the nth year, where $S=350\left(1.07+1.07^2+\ldots+1.07^n\right)$. By using the sum of a geometric series, find the least number of years necessary for the total in the scheme to exceed £25 000.

2. Express $u_r=\frac{1}{(2r-1)2r(2r+1)}$ in partial fractions.

   Denoting $\sum_{r=1}^{n} u_r$ by S_n, prove that

   $$S_n=\frac{1}{2(2n+1)}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots+\frac{1}{2n-1}-\frac{1}{2n}.$$

   By considering the Maclaurin series for $\ln(1+x)$, or otherwise, find $\lim_{n\to\infty} S_n$.

3. Write down and simplify the first three terms in the series expansion of $\left(1+\frac{x}{3}\right)^{-\frac{1}{2}}$ in ascending powers of x. State the set of values of x for which the series is valid.

   Given that x is so small that terms in x^3 and higher powers of x may be neglected, show that

   $$e^{-x}\left(1+\frac{x}{2}\right)^{-\frac{1}{2}}=1-\frac{7}{6}x+\frac{17}{24}x^2.$$

4. Prove by mathematical induction that, for all positive integers n,

$$\sum_{r=1}^{n}\frac{3r+2}{r(r+1)(r+2)}=\frac{n(2n+3)}{(n+1)(n+2)}$$

Hence, or otherwise, find the sum of the infinite series

$$\frac{8}{2\times3\times4}+\frac{11}{3\times4\times5}+\frac{14}{4\times5\times6}+\ldots+\frac{3k+2}{k(k+1)(k+2)}+\ldots$$

5. (a) Show that $(2k+1)^4-(2k-1)^4\equiv 64k^3+16k$.
Using this identity, prove that

$$\sum_{k=1}^{n}\left(64k^3+16k\right)=(2n+1)^4-1$$

(b) Assuming the result $\sum_{k=1}^{n}k=\frac{1}{2}n(n+1)$, use the result in (a) to prove that $\sum_{k=1}^{n}k^3=\frac{1}{4}n^2(n+1)^2$.

(Oxford)

6. A geometric series has first term 1 and common ratio $\frac{1}{2}\sin 2\theta$.

(a) Find the sum of the first 10 terms in the case when $\theta=\frac{\pi}{4}$, giving your answer to 3 decimal places.

(b) Given that the sum-to-infinity is $\frac{4}{3}$, find the general solution for θ in radians.

7. (a) Use the formulae for $\sum_{r=1}^{n}r$ and $\sum_{r=1}^{n}r^2$ to show that $\sum_{r=0}^{n}(r+1)(r+2)=\frac{1}{3}(n+1)(n+2)(n+3)$.

(b) Using partial fractions, or otherwise, find the sum of the series $S_n=\sum_{r=0}^{n}\frac{1}{(r+1)(r+2)}$.

Deduce the value of $\lim_{n\to\infty}S_n$.

8. Given that $e^y=e^x+e^{-x}$, show that $\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2-1=0$. Find the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x=0$. Hence determine the Maclaurin series for y in ascending powers of x up to and including the term in x^4.

9. All the terms of a certain geometric progression are positive. The first term is a and the second term is a^2-a. Find the set of values of a for which the series converges.

Given that $a=\frac{5}{3}$:

(a) find the sum of the first 10 terms of the series, giving your answer to 2 decimal places;

(b) show that the sum-to-infinity of the series is 5;

(c) find the least number of terms of the series required to make their sum exceed 4.999.

10. Use induction to prove that

$$\sum_{r=1}^{n}r^2=\frac{1}{6}n(n+1)(2n+1)$$

Hence find $\sum_{r=1}^{n}(n+r-1)(n+r)$, giving your answer in terms of n.

11. (a) Write down in ascending powers of x the series for $\ln(1-x)$, where $|x|<1$. Hence find $\sum_{r=1}^{\infty}\frac{1}{r\times2^r}$, giving your answer in the form $\ln p$, for some number p.

(b) Find $\sum_{r=1}^{\infty}\frac{1}{(r+1)!2^r}$.

12. Given that $|x|<\sqrt{3}$, determine the expansion of $f(x)=\frac{16-x}{(2-x)\left(3+x^2\right)}$ in ascending powers of x up to and including the x^3 term. Hence find, correct to four decimal places, an approximation to

$$\int_0^1\frac{16-x}{(2-x)\left(3+x^2\right)}dx$$

13. By considering $\sum_{n=1}^{N}\left[\cos\left(n-\frac{1}{2}\right)x-\cos\left(n+\frac{1}{2}\right)x\right]$, or otherwise, show that

$$\sum_{n=1}^{N}\sin nx=\operatorname{cosec}\frac{1}{2}x\times\sin\left[\frac{1}{2}(N+1)x\right]\times\sin\left(\frac{1}{2}Nx\right),$$

provided that $\sin\frac{1}{2}x\neq0$. Deduce that

$\sum_{n=1}^{N-1}\sin\frac{n\pi}{N}<\operatorname{cosec}\frac{\pi}{2N}$ for all $N\geq2$. (Cambridge)

14. (a) Write down and simplify binomial series in ascending powers of x up to and including the terms in x^3 for $(1+x)^{-1}$, $(1-2x)^{-1}$ and $(1-2x)^{-2}$.

(b) Express $\dfrac{6-11x+10x^2}{(1+x)(1-2x)^2}$ in partial fractions.

(c) Using the series from (a), expand $\dfrac{6-11x+10x^2}{(1+x)(1-2x)^2}$ in ascending powers of x up to and including the term in x^3.

15. Prove by induction that $\sum_{r=1}^{n} r(3r+1) = n(n+1)^2$ for all positive integers n. (Cambridge)

16. Show that the sum of the first 50 terms of the geometric series with first term 1 and common ratio $r = 2^{\frac{1}{50}}$ is $\dfrac{1}{r-1}$. Hence, by using the trapezium rule with 50 intervals of equal width, show that $\int_0^1 2^x dx$ is approximately $\dfrac{r+1}{100(r-1)}$, where $r = 2^{\frac{1}{50}}$. Evaluate this expression to 3 decimal places.

17. Write down the expansions of
(i) $\ln(1+2x)$, (ii) $\ln(1+3x)$, (iii) $(1-2x)^{-3}$
in ascending powers of x up to and including the term in x^3. Given that x is small enough for x^4 and higher powers of x to be ignored, show that

(a) $\ln\left\{\dfrac{(1+2x)^3}{(1+3x)^2}\right\} = 3x^2 - 10x^3$

(b) $(1-2x)^{-3} - e^{6x} - 2\ln\left\{\dfrac{(1+2x)^3}{(1+3x)^2}\right\} = kx^3$ for some constant k, stating the value of k.

18. (a) By considering the expansion of $\ln(1-x)$ find the sum of the infinite series $\sum_{r=1}^{\infty} \dfrac{1}{r \times 2^{r-1}}$.

(b) Write down the first four terms and the general term of the expansion in ascending powers of x of $(1-x)^{-1}$. Hence, by differentiation, obtain the first 3 terms and the general term in the expansion of $(1-x)^{-2}$, and find the sum of the infinite series

$$\sum_{r=1}^{\infty} \frac{r}{2^{r-1}}.$$

19. Assuming the series expansions for $\cos x$, $\ln(1+x)$ and $(1+x)^{-1}$, show that the series expansions of $\cos[\ln(1+x)]$ and $\sec[\ln(1+x)]$ are given respectively by $1 - \frac{1}{2}x^2 + \frac{1}{2}x^3 - \frac{5}{12}x^4 + \ldots$ and $1 + \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{2}{3}x^4 + \ldots$
Show by differentiating the series for $\cos[\ln(1+x)]$ that the series for $\sin[\ln(1+x)]$ is $x - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots$ and use this together with the series for $\sec[\ln(1+x)]$ to find a series expansion for $\tan[\ln(1+x)]$ as far as the x^3 term.

20. Let $\phi(j) = \dfrac{4^{j+1}(j-1)}{3(j+2)}$

(a) Show that $\phi(j) - \phi(j-1) = \dfrac{j^2 4^j}{(j+1)(j+2)}$

(b) Hence find, in terms of n, the sum of the series $\sum_{j=1}^{n} \dfrac{j^2 4^j}{(j+1)(j+2)}$ (Cambridge)

21. Given that $f(x) = (2x-1)(2x+1)$,

(a) express $\sum_{r=1}^{n} f(r)$ as a cubic in n

(b) show that $\sum_{r=1}^{n} \dfrac{1}{f(r)} = \dfrac{n}{2n+1}$

(c) evaluate $\lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{f(r)}$

(d) find the coefficient of x^n when $\dfrac{1}{f(x)}$ is expanded in ascending powers of x, distinguishing between the cases n odd and n even. State the set of values of x for which this expansion is valid.