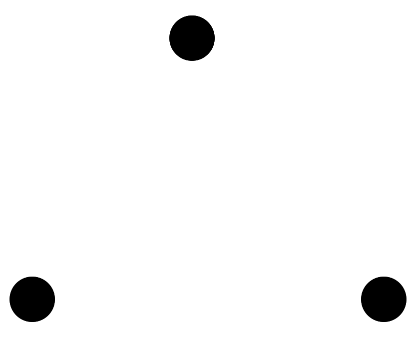
Math 215 - Project Number 1 -- Graph Theory and The Game of Sprouts

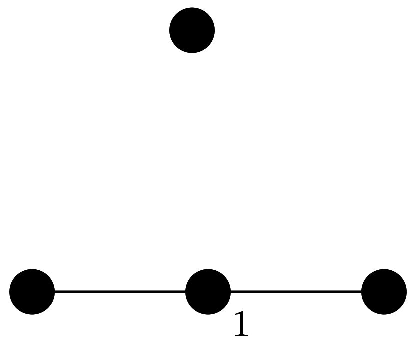
This project introduces you to some aspects of graph theory via a game played by drawing graphs on a sheet of paper. The game is called "sprouts" and it is an invention of John Horton Conway. In the course of the problems we shall also work on writing proofs that use mathematical induction. The purpose of this part of the course is to give you some practice exploring a mathematical domain and seeing both results and proofs emerging naturally from questions and explorations.

Proof by induction is an important technique that applies to all areas of mathematics.

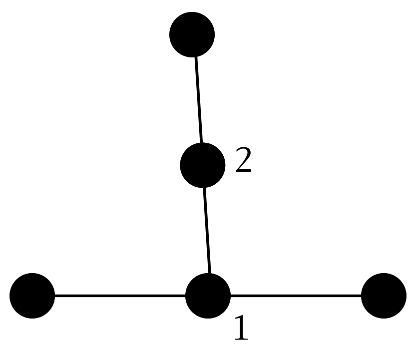
Sprouts is a game played by two players, using a sheet of paper and a pen or a pencil. The game begins with a choice by the players of a collection of "spots" or "nodes" on the paper. We illustrate here with three spots.



A move in Sprouts is accomplished by connecting two spots with an edge and placing a new spot in the middle of the edge. Below you see the result of the first player connecting two spots and placing the new spot.



We have labeled the new spot with the number 1 just so we can remember who did the move. Now the second player makes a move.

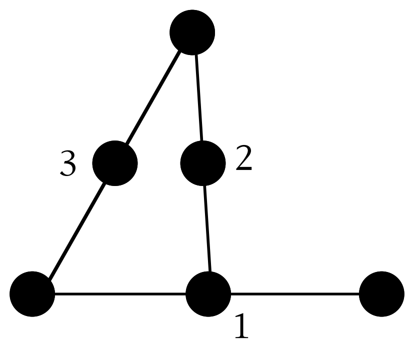


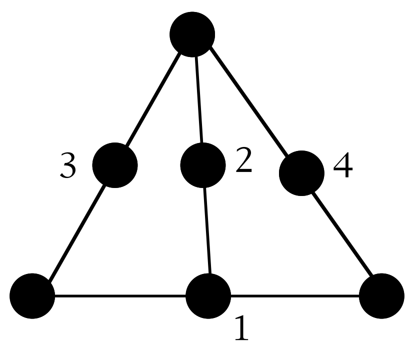
Notice that you can tell that the second new spot is due to the second player since it is labeled by 2. The real point, if we continue labeling in this way, is that second player will always label with even numbers and first player will always label with odd numbers. Lets call the players **First** and **Second.**

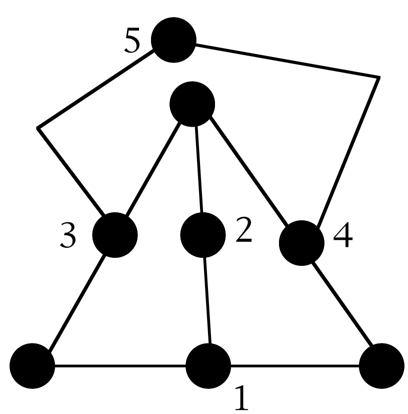
I have to tell you one more thing about this game:
When a spot has three edges attached to it (locally) then it is COMPLETE. If a spot is complete, you can no longer connect any further edges to that spot. Thus the spot labeled 1 in the above figure is now complete.

But what is the goal in this game? I am glad you asked! *The first person to make a move so that his opponent can not make a reply is the winner.*

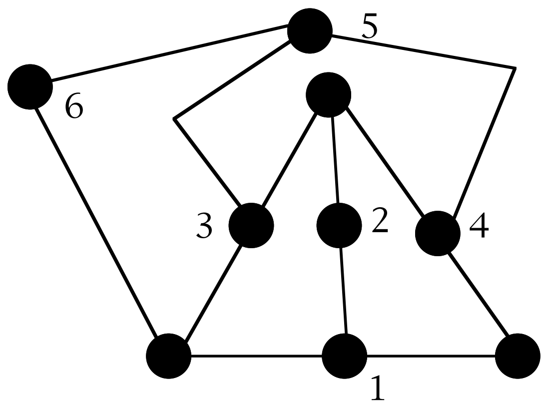
Lets follow this game for a bit and see who wins.

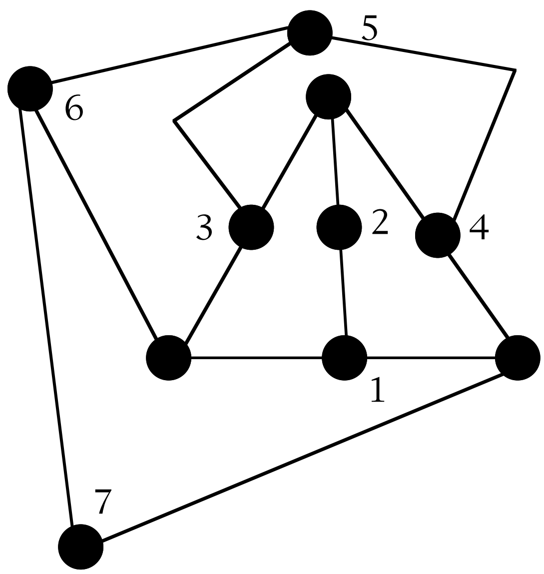






In this last position we have drawn an "edge with angles" between 3 and 4 and placed spot 5 on the middle of the angular edge. If you were drawing with a pen or a pencil you would just draw a curved edge connecting 3 and 4, but I have been using a simple drawing program that only makes straight segments. The restrictions on the edges is that the **new edges touch the original graph only at the end-spots. You are not allowed to have one edge cross through another one.**





The game is over. Even though spot s 2 and 7 are incomplete, all the other spots are complete, and there are no moves left. The game is won by the player who put in spot 7. This was the First Player since 7 is odd. Notice that the last diagram contains a complete record of the game.

Problem 1.1. Find an opponent and play a number of games of 2-spot and 3-spot sprouts. Find out the best strategy that you can for 2-spot sprouts, and write a short essay describing your finding. This essay should fit on one side of a sheet of paper.
Due Friday, January 16, 2009.

Problem 1.2. Prove that every game of sprouts starting with any finite number of spots (say with N spots) must eventually end, no matter how the players play. (Of course they must obey the rules.)

If the game starts with N spots, give an upper bound on the number of moves in any game.
Due Wednesday, January 21, 2009.

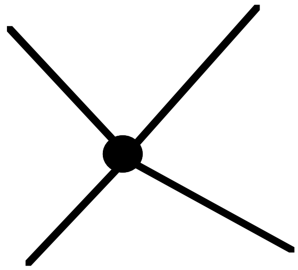
Problem 1.3. Read pages 39 - 51 in our text "An Introduction to Mathematical Reasoning" and do problems 5.1 to 5.4 on page 51.
Due Monday, January 26, 2009.

Problem 1.4. Prove by induction that
(1/1x3) + 1/(3 x 5) + 1/(5 x 7) + ... + 1/((2n-1) x (2n + 1)) = n/(2n + 1).
Due Monday, January 26, 2009.

Problem 1.5. The Fibonacci Numbers are the numbers in the sequence 1,1,2,3,5,8,13,21,34,55,144,189,....
We denote these by $f_1 = 1$, $f_2 = 1$, $f_3 = 3$, $f_4 = 5$ and
$f_{n+1} = f_n + f_{n-1}$.
Each Fibonacci number is the sum of the previous two Fibonacci numbers. There are many patterns in the Fibonacci series.
Notice that
$1^2 = 1 = 1 \times 1$
$1^2 + 1^2 = 2 = 1 \times 2$
$1^2 + 1^2 + 2^2 = 6 = 2 \times 3$
$1^2 + 1^2 + 2^2 + 3^2 = 15 = 3 \times 5$.
Prove by induction that
$f_1^2 + f_2^2 + ... + f_n^2 = f_n \times f_{n+1}$ for all n=1,2,3,....
Due Friday, January 30,2009.

Problem 1.6. (Returning to Sprouts) Prove that for every natural numbrer n (n = 1, 2, 3, 4, ...) there is a sprouts game, starting with n sprouts, that ends in exaactly 3n -1 moves. You can use mathematical induction in your proof.
Due Monday, February 2, 2009.

Problem 1.7. In *4-sprouts* we use the same rules as in ordinary sprouts, but we allow 4 lines to touch a spot
as in the diagram below.



A move has the same form as in regular spots, but notice that the new spot, having two lines going into it, has two freedoms in the game of 4-spots. Show that some games of 4-spots can go on forever. Give specific examples. Think about the question of how to modify the rules of 4-spots so that it will become a game that always ends in a finite number of moves.
Due Wednesday, February 4, 2009.

Problem 1.8. The Fibonacci Numbers are the numbers in the sequence 1,1,2,3,5,8,13,21,34,55,144,189,....
We denote these by $f_1 = 1$, $f_2 = 1$, $f_3 = 3$, $f_4 = 5$ and $f_{n+1} = f_n + f_{n-1}$. We also take $f_0 = 0$ by convention.
Prove the following formula by induction.

$$f_{n-1} \times f_{n+1} = f_n^2 + (-1)^n \text{ for } n = 1,2,\ldots$$

For example $8 \times 21 = 168 = 169 - 1 = 13^2 - 1$.
Due Friday, February 6,2009.

Problem 1.9. Problem 20. page 56 of our textbook.
Due, Monday, February 9,2009.

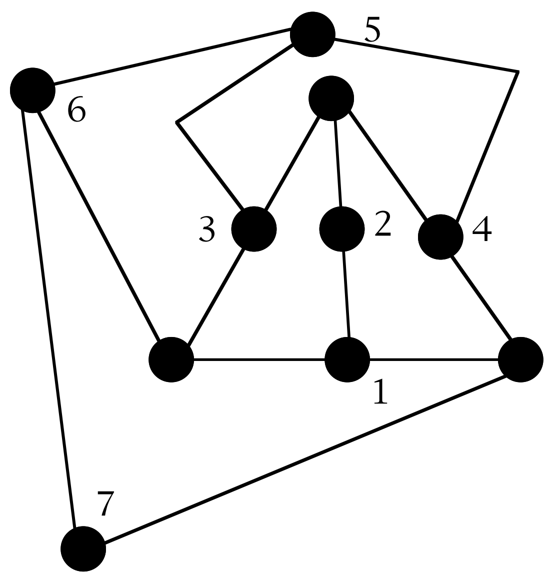
Problem 1.10. Prove by induction that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$$

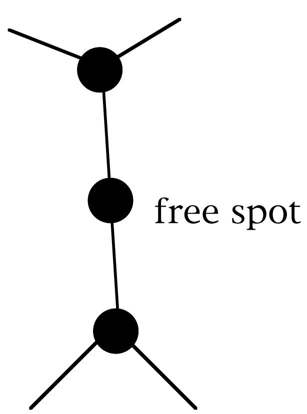
for n = 1,2,3,... .
Due, Monday, February 9,2009.

Sprouts Endgame - Counting Moves and Pharisees.
Lets think about endgame positions in sprouts. We know that there must be some spots that have exactly one freedom. For example, in the game below we have n = 3 as the number of starting spots, and at the endgame position there are the spots labelled 2 and 7 each with one freedom. Note that the number of moves in this game is 7 and that $3 \times n - 7 = 9 - 7 = 2$. This illustrates a general feature of an endgame position. The number of free spots (spots with one freedom) is equal, at the engame to 3n - m where m is the number

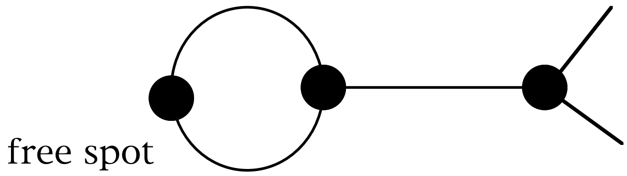
of moves and n is the number of starting spots. If we denote by F the number of free spots at the endgame, then the result is $F = 3n - m$. Theproof is simply that each move reduces the number of freedoms by 1; the starting number of freedoms is 3n; and all the freedoms at the end are concentrated in free spots (with one freedom each).



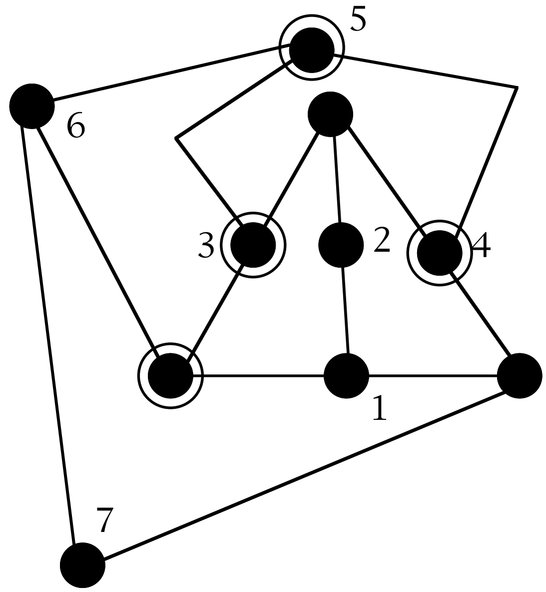
Now lets go a bit further and look at the completed spots near a free spot. In our example above these have the form shown below.



There are two neighboring spots, both with no freedoms (else we would not be at the end of the game). Another way this can happen is illustrated below. Again there are two nearby competed spots that can be regarded as the neighbors of the free spot.



Each free spot at the end of the game has two unique neighbors. *These neighbors are neighbors only of that free spot.* This last statement requires proof. We see the truth of it by enumerating the cases of possible shared neighbors. We leave the proof of this unique neighbors statement to you as an excercise (but you do not have to hand this one in as an assigment). Now using the uniqueness of neighbors, we see that if N is the total number of neighbors in an endgame position, then $N = 2F$ where F is the number of free spots.
And the total number of spots at the end of the game is of course $n + m$. Therefore $n+m = F + 2F + P = 3F + P$ where *P is the number of spots at the end that are neither free nor neighbors of free spots. We call such spots Pharisees.* P is the number of Phaisees. For example, in the example below, the spots with the extra circles around them are the Pharisees.



We can count. We have
$F = 3n - m$
and

$n + m = 3F + P.$
Therefore
$n + m = 3(3n - m) + P$
$n + m = 9n - 3m + P$
$4m = 8n + P$
$m = 2n + P/4.$
It is this last equation that is significant.
We have proved the following

Theorem. *In a sprouts game that lasts m moves and starts with n spots, the number of moves is related to the number P of Pharisees in the endgame position by the formula.*

$$m = 2n + P/4.$$

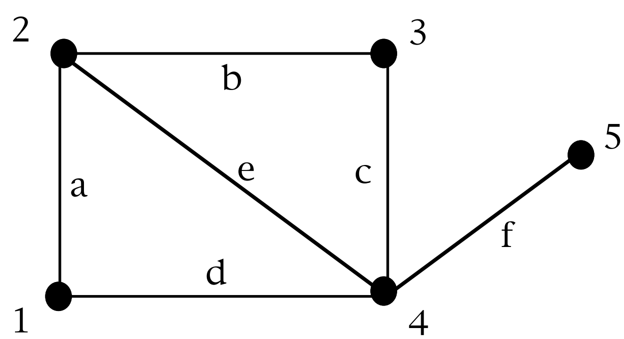
In particular this means that no sprouts game can end in fewer than $2n$ moves, and the number of Pharisees is always divisible by 4.

Problem 1.11. Suppose that you play a sprouts game with n starting spots and that the game lasts for $3n - 1$ moves. How many Pharisees will there be at the end of the game? Illustrate your result with a specific example.
Due, Wednesday, February 11,2009.

##

Graph Theory and Euler's Formula.

A *graph* is a collection of *nodes* or *vertices*, usually depicted as dark spots or points, and a collection of *edges* that can connect two nodes or connect a node with itself. For example, the graph below has five nodes and six edges. It is a c*onnected* graph in the sense that there is a pathway along the edges between any two nodes.



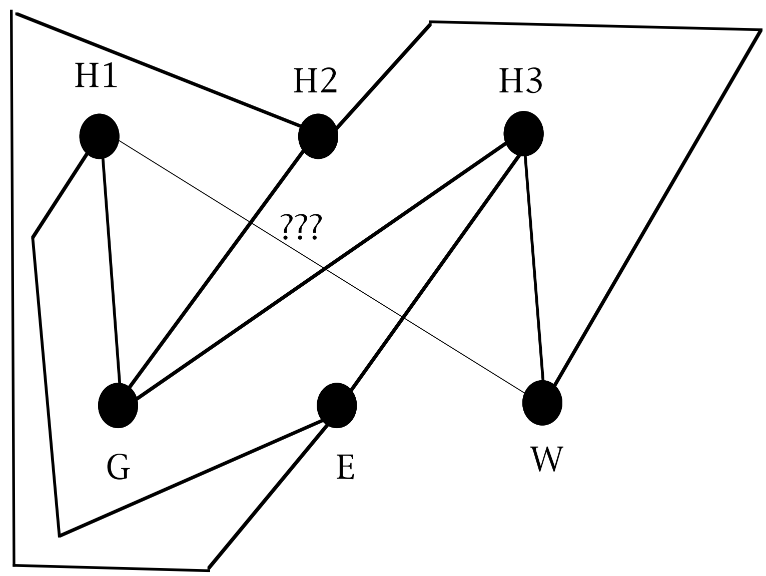
A connected graph G.

Graphs are fundamental mathematical structures and they have lots of applications. We are all familiar with the graphical notation for electrical circuits. Subway system maps are graphs with special decorations. In general, when we want to desribe engineering systems, economic systems, and other systems of relationship, we can start with a collection of definite entities (the nodes) and the information about how they are connected with one another (the edges).

As you can see, the game of sprouts is a game that is played by constructing a graph. The graph constructed in sprouts is special in that it cannot have more than three edges touching any node, and the sprouts graph is drawn in the plane in such a way that no two edges of it touch except at the nodes of th graph. We say that a graph that can be drawn in the plane in this way is a **plane graph**. The graph in the figure above is also a plane graph.

Not every graph is planar! That is if you specify a set of nodes and a set of connections to be made among these nodes, it may not be possible to accomplish these connections in the plane without having some edges cross over one another. Here is a problem that will show you how that can happen.

Problem 1.12. (The Gas-Electricity-Water Problem)
Three companies, the gas company, the electricity compay and the water company want to make connections from the gas main (G), the electrical source (E) and the water main (W) to three houses (H1, H2 and H3). They wish to lay their lines so that no two lines meet except at the sources (G,E and W) and at the houses (H1, H2 and H3). Can you find a solution to this design problem? If not, then why not?



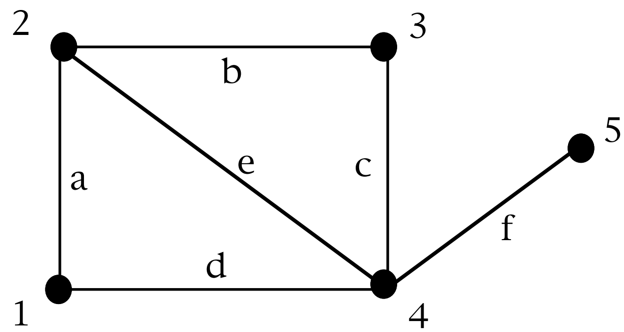
In the illustration above the city planners have drawn a graph to help them design the connections but they have run into a difficulty with making a water line from W to H1. Everything went fine with the design up to that point, but then there does not seem to be any way to conncet from W to H1 without crossing previously created lines. It will cost the city a great deal to dig tunnels to make lines cross over one another. So these designers really need to know whether the job can be done with no crossovers, and if it cannot be done that way, then they want to know the least number of crossovers that are needed to do the job.

Due, Wednesday, February 11,2009.

We will now discuss a formula about plane graphs that was discoverd around 1750 by the Swiss mathematician Leonhard Euler.
<http://en.wikipedia.org/wiki/Leonhard_Euler>
Euler was one of the greatest mathematicians of all time, and his formula about plane graphs is the beginning of the subjects of graph theory and topology (topology is the study of mathematical spaces and includes and generalizes classical geometry). Here is Euler's result:

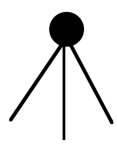
Theorem. Let **G** be a connected finite plane graph with **V** nodes, **E** edges and **F** faces (a face is a region in the plane that is delineated by the graph in the plane). Then $V - E + F = 2$.

Here is an example of Euler's formula for a specific graph in the plane.



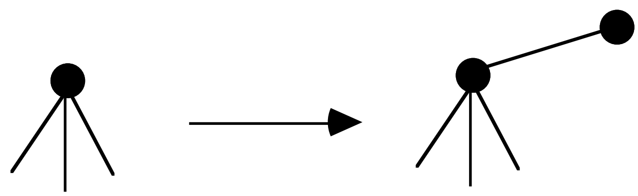
Here we have V = 5, E = 6 and F = 3. The regions we count are the interiors of the two triangles and the outer region consisting in the rest of the plane. Note that V - E + F = 5 - 6 + 3 = 2 as promised by Euler's Theorem.

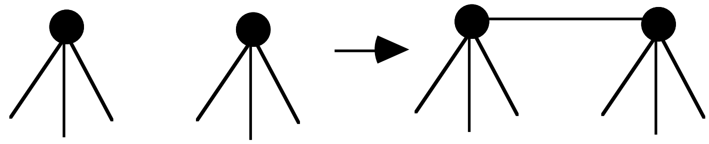
Problem 1.13. Construct a proof of Euler's formula by induction on the total number of edges and vertices in the graph **G**. You should consider how the graph can be built up from simpler graphs by adding edges to them. In fact, any connected graph can be built from a single vertex graph by adding new edges in two ways that I will now explain, but first we introduce an abbreviation: The diagram below stands for s*ome vertex in a larger graph*.



You can tell when I am using this abbreviation because the edges that go out of this vertex are not meeting any other vertices in the picuture. The picture is a shorthand for a possibly larger and more complete picture. In the abbreviation we show three edges touching the vertex. In a real situation some edges touch the vertex, but the number is not necessarily equal to three. Ok?

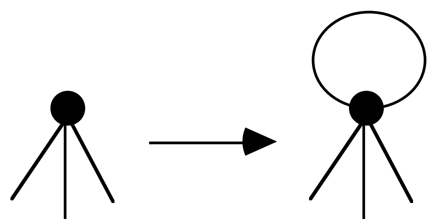
Now lets use this and illustrate two ways to make a larger graph.

I. 

II. 

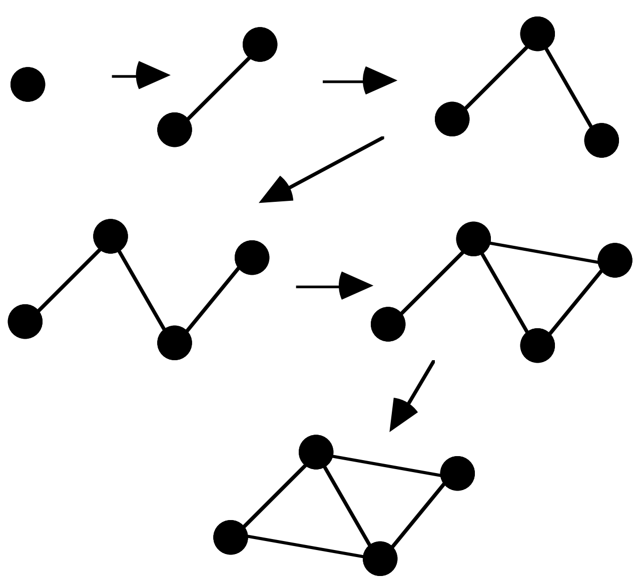
In method number I we add a new edge and a new vertex by attaching the new edge to an already existing vertex. In method number II we connect two vertices with a new edge.

Remark. We regard the move



as a special case of II.

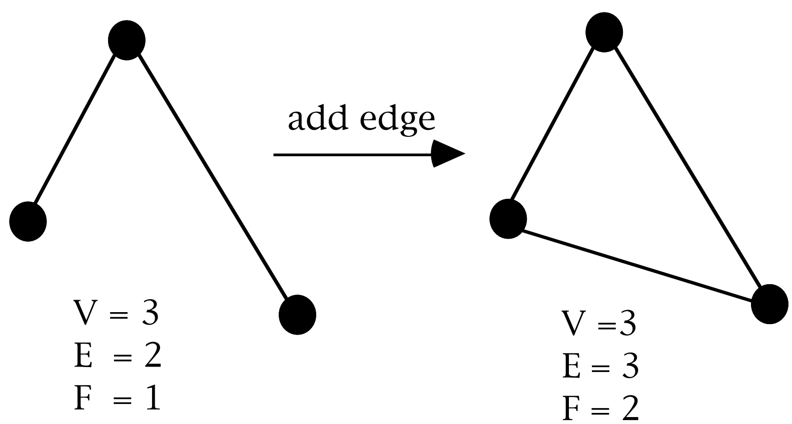
I claim that *any connected graph can be built up by performing a sequence of operations of these two types*. Here is an example.



You can use this claim in your proof, and if you want, you can also make a proof of the claim. We will discuss why the claim is true in class.

Now, to prove the Euler Theorem, you can proceed by induction, showing that V - E + F does not change its value when you perform a move of type I or type II. You will find that it is very easy to see this for type I, and that in order to see it for type II you need to start with a connected graph. If the graph is connected, then a move of type II will create a new region in the graph. Look at the example above and see how this works. You can use this fact also in your proof (that a move of type II will create a new region). You should then be able to construct an inductive proof of the Euler formula.

Here is an example:
We create a triangle graph by adding an edge to a tree.



Note that adding the edge creates a new region, and V-E + F does not change from before to after the addition of the new edge.

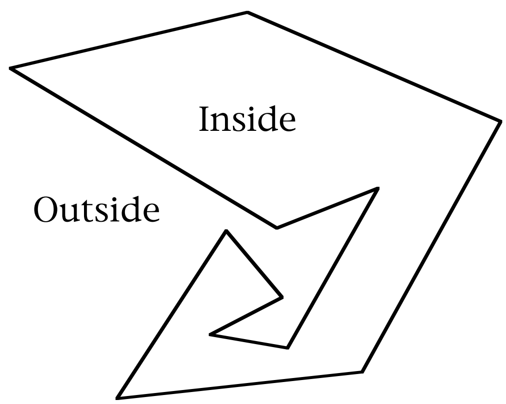
(c) Discuss your proof of the Euler formula with another student in the class. Do you both feel that the proof is complete? What might be missing? In this problem, it worth having the discussion. We will discuss some of the issues related to the problem in class before the problem is due.
Due, Friday, February 20,2009.

Supplement to Problem 1.13.

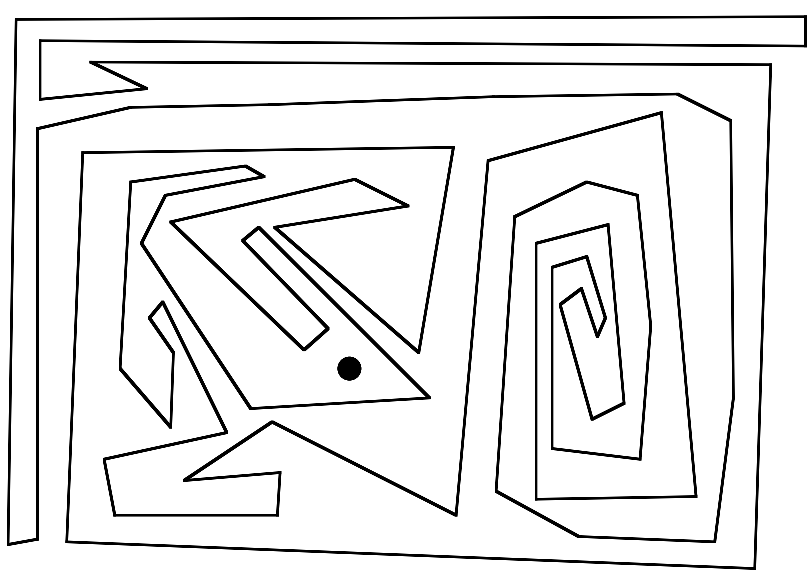
There is a fact about curves in the plane that you can use in thinking about regions that are created when graphs are drawn in the plane.
This fact is called the

Jordan Curve Theorem: *A closed curve in the plane without any self-intersections divides the plane into exactly two regions.*

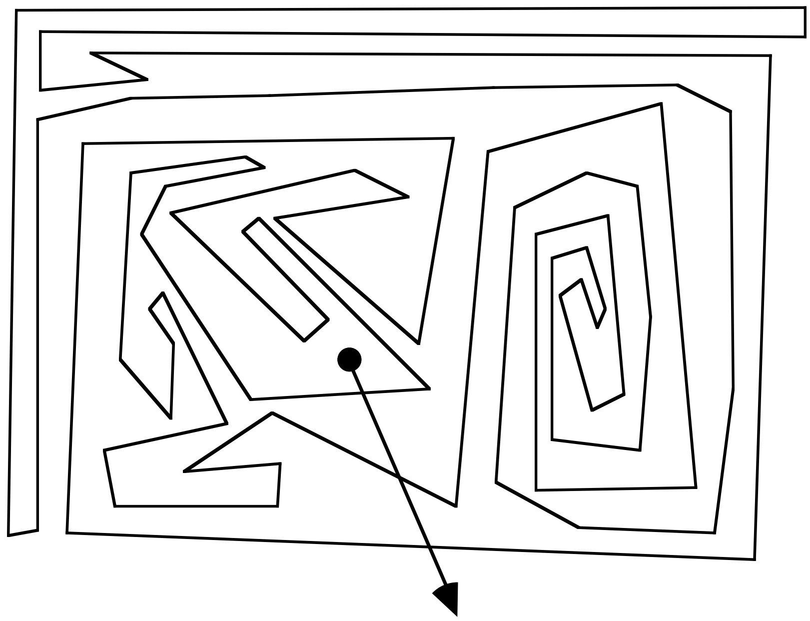
Here is an example:



You are not required to prove this result, but you can use it and it is interesting to see how complex examples can look!



Is the black dot inside or outside this curve?
Of course you can solve this like solving a maze, but look!



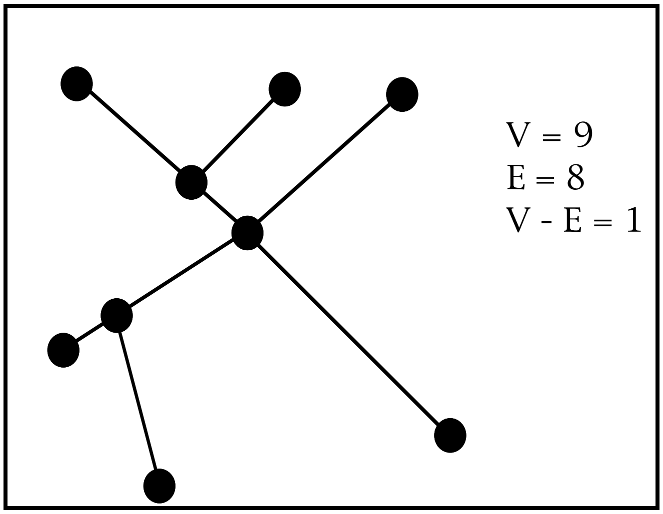
An arrow from the dot intesects the curve in an ODD number of points. I claim that this tells you that the point must be INSIDE. If the intersection number were EVEN, then the point would be outside. Can you explain why this works? (I say explain, and of course I am hoping that your explanation will turn into a mathematical proof. But lets explore.)

We will discuss in class why and how the Jordan Curve Theorem is relevant to proving Euler's Formula.

Remark. Another approach to the Euler formula uses the concept of a tree: A graph is said to be a *tree* if it does *not* contain any cycles (a cycle is a sequence of distinct edges such that the each edge shares its endpoints with the edges before and after it in the sequence. For example in the graph above, bce is a cycle and abcd is a cycle. When a plane graph has no cycles then the only region it can delineate is the rest of the plane other than itself, and so a tree has $F = 1$.

Show that for a connected tree, $V - E = 1$.

From this is follows that for connected plane trees $V - E + F = 2$, and so we know the Euler formula already for trees.



The picture above illustrates this result for trees. You can prove that V - E = 1 for a connected tree by induction on the number of edges in the tree.

You can then prove the Euler formula for an arbitrary connected plane graph by just making that graph by adding edges by our type II move to a tree. Think about this and try some examples.