Squeeze Theorem Examples

Squeeze Theorem. *If*

$$f(x) \leq g(x) \leq h(x)$$

when x is near a (but not necessarily at a [for instance, $g(a)$ may be undefined]) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L,$$

then

$$\lim_{x \to a} g(x) = L$$

also.

Example 1. Find

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right).$$

When trying to find functions to use to 'squeeze' $g(x)$, we want functions that are, a) similar enough to $g(x)$ that we can be sure the squeeze works, b) easier to evaluate their limit as $x \to a$. We typically do this by starting with the most complicated or troublesome part of $g(x)$, see if we can find constants (or simpler functions) that it stays between, and then multiply in the rest of 'nicer' parts of $g(x)$.
In this case, the part of $g(x)$ that is giving us the most trouble is the $\cos\left(\frac{1}{x^2}\right)$ part (we get division by 0 if we try direct substitution). Now we know that cosine stays between -1 and 1, so

$$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$$

for any x in the domain of the function (i.e., any $x \neq 0$). Since x^2 is always positive, we can multiply this inequality through by x^2:

$$-x^2 \leq x^2 \cos\left(\frac{1}{x^2}\right) \leq x^2$$

So, our original function is bounded by $-x^2$ and x^2. Now since

$$\lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0,$$

then, by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0.$$

Example 2. Find

$$\lim_{x \to 0} x^2 e^{\sin\left(\frac{1}{x}\right)}.$$

As in the last example, the issue comes from the division by 0 in the trig term. Now the range of sine is also $[-1,\ 1]$, so

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Taking e raised to both sides of an inequality does not change the inequality, so

$$e^{-1} \leq e^{\sin\left(\frac{1}{x}\right)} \leq e^1,$$

and once again we can multiply through by x^2 and get

$$x^2 e^{-1} \leq x^2 e^{\sin\left(\frac{1}{x}\right)} \leq x^2 e^1.$$

So, our original function is bounded by $e^{-1}x^2$ and ex^2, and since

$$\lim_{x \to 0} e^{-1}x^2 = \lim_{x \to 0} ex^2 = 0,$$

then, by the Squeeze Theorem,

$$\lim_{x \to 0} x^2 e^{\sin\left(\frac{1}{x}\right)} = 0.$$